A new spectral sequence for fusion systems

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Computations in Groups and Algebras (CoGrAl2015)
16th - 19th February, 2015, Jena (Germany)
Friedrich-Schiller-Universität Jena
Fakultät für Mathematik und Informatik
Institut für Mathematik
Arbeitsgruppe Algebra/Zahlentheorie

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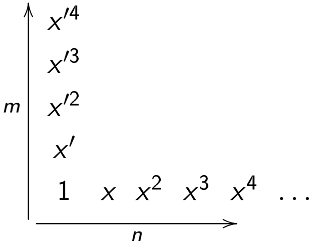

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m						
	x'^4	xx'^4	$x^2x'^4$	$x^3x'^4$	$x^4x'^4$	$\dots$
	x'^3	xx'^3	$x^2x'^3$	$x^3x'^3$	$x^4x'^3$	$\dots$
	x'^2	xx'^2	$x^2x'^2$	$x^3x'^2$	$x^4x'^2$	$\dots$
	x'	xx'	x^2x'	x^3x'	x^4x'	$\dots$
	1	x	x^2	x^3	x^4	$\dots$

n

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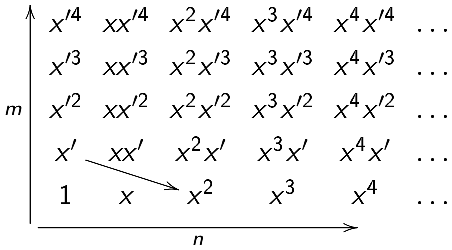

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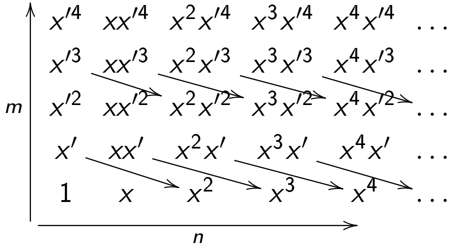
$$d_2(x^i x') = x^{i+2}$$

$$d_2(x'^2) = 2d_2(x') = 0$$



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$E_3^{n,m}$

	x'^4	xx'^4	0	0	0	...
	0	0	0	0	0	...
m	x'^2	xx'^2	0	0	0	...
	0	0	0	0	0	...
	1	x	0	0	0	...

n

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$$H^*(C_4; \mathbb{F}_2) = \Lambda(x) \otimes \mathbb{F}_2[y]$$

$$|x| = 1,\ x^2 = 0,\ |y| = 2,\ y = x'^2$$

$$m \uparrow \begin{array}{cccccc} x'^4 & xx'^4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ x'^2 & xx'^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & x & 0 & 0 & 0 & \dots \end{array} \quad \xrightarrow{n}$$

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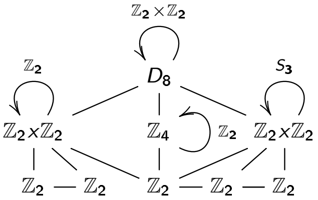

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The E_2-page are the $\mathcal{F}$-invariants in the second page $H^p(S/T; H^q(T; \mathbb{F}_p))$ of the Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence of p-groups

$$T \to S \to S/T.$$

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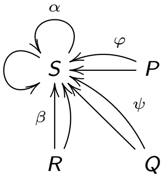

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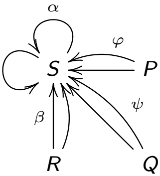
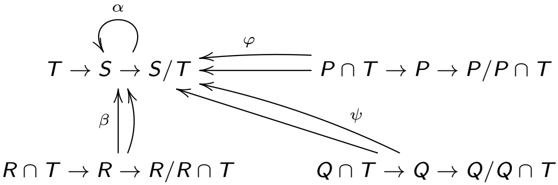
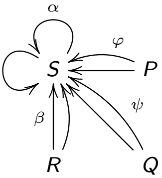
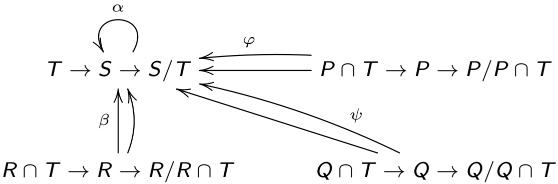


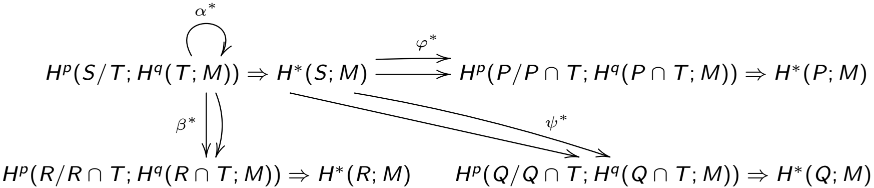




α
φ
S
P
ψ
β
R
Q



α
φ
T → S → S/T
P ∩ T → P → P/P ∩ T
β
ψ
R ∩ T → R → R/R ∩ T
Q ∩ T → Q → Q/Q ∩ T



α*
φ*
H^p(S/T; H^q(T; M)) ⇒ H*(S; M)
H^p(P/P ∩ T; H^q(P ∩ T; M)) ⇒ H*(P; M)
β*
ψ*
H^p(R/R ∩ T; H^q(R ∩ T; M)) ⇒ H*(R; M)
H^p(Q/Q ∩ T; H^q(Q ∩ T; M)) ⇒ H*(Q; M)

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1. $U_3(2^n), Sz(2^n),\ p = 2,$
2. Groups of Lie type in characteristic $\neq p$ whose Sylow p-subgroup is abelian but not elementary abelian (p odd).
3. $U_3(3^{2n+1}),\ p = 3.$
4. $Re(3^{2n+1}),\ n \geq 1,\ p = 3.$
5. $G_2(q),\ (q, 3) = 1,\ p = 3.$
6. $J_2, J_3,\ p = 3.$
7. $Co_2, Co_3, HS, Mc,\ p = 5.$
8. $J_4,\ p = 11.$

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Cohomology of J_2 over $\mathbb{F}_3$

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- $E_2(LHS(\mathbb{Z}_3 \to 3_+^{1+2} \to \mathbb{Z}_3 \times \mathbb{Z}_3))^{C_8} \Rightarrow H^*(J_2; \mathbb{F}_3) \leq H^*(S; \mathbb{F}_3)$

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To compute $H^*(J_2;\mathbb{F}_3)$ we use a classical approach: First, we determine the Poincaré series of this ring from its associated bigraded algebra $E_6^{C_8} = E_\infty^{C_8}$. Then we find generators for this bigraded algebra and lift them to $H^*(S;\mathbb{F}_3)$. Finally, we find relations in the bigraded algebra and lift them, using as stopping criterion the Poincaré series.

Cohomology of J_2 over $\mathbb{F}_3$

$E_2(LHS(\mathbb{Z}_3 \to 3^{1+2}_{+} \to \mathbb{Z}_3 \times \mathbb{Z}_3))$:

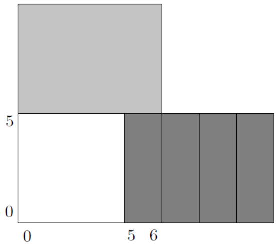
2	t	ty_1, ty_2	tx_1, tx_2 ty_1y_2	ty_1x_1, ty_1x_2 ty_2x_1, ty_2x_2
1	u	uy_1, uy_2	uy_1y_2 uy_1y_2	uy_1x_1, uy_1x_2 uy_2x_1, uy_2x_2
0	1	y_1, y_2	x_1, x_2 y_1y_2	y_1x_1, y_1x_2 y_2x_1, y_2x_2
	0	1	2	3

The extension $\mathbb{Z}_3 \to 3^{1+2}_{+} \to \mathbb{Z}_3 \times \mathbb{Z}_3$ is central.

Cohomology of J_2 over $\mathbb{F}_3$

$E_6(LHS(\mathbb{Z}_3 \to 3_+^{1+2} \to \mathbb{Z}_3 \times \mathbb{Z}_3))$:

5							
4							
3			uty_1y_2				
2		ty_1, ty_2		ty_1x_1, ty_1x_2 ty_2x_2		$tx_1^2y_1, tx_1^2y_2$ $tx_2^2y_1, tx_2^2y_2$	
1		uy_1, uy_2	uy_1y_2	uy_1x_1, uy_1x_2 uy_2x_1, uy_2x_2		$ux_1^2y_1, ux_1^2y_2$ $ux_2^2y_1, ux_2^2y_2$	
0	1	y_1, y_2	x_1, x_2	y_1x_1, y_1x_2 y_2x_2	x_1^2, x_2^2 x_1x_2	$x_1^2y_1, x_1^2y_2$ $x_2^2y_1, x_2^2y_2$	x_1^3, x_2^3 $x_1^2x_2, x_1x_2^2$
	0	1	2	3	4	5	6



There are both vertical and horizontal periodicities: $E_6^{n,m} \cong E_6^{n,m+6}$ for $n, m \geq 0$ and $E_6^{n,m} \cong E_6^{n+2,m}$ for $n \geq 5$ and $m \geq 0$.

$E_6(LHS(\mathbb{Z}_3 \to 3_+^{1+2} \to \mathbb{Z}_3 \times \mathbb{Z}_3))$:

5							
4							
3			uty_1y_2				
2		ty_1, ty_2		ty_1x_1, ty_1x_2 ty_2x_2		$tx_1^2y_1, tx_1^2y_2$ $tx_2^2y_1, tx_2^2y_2$	
1		uy_1, uy_2	uy_1y_2	uy_1x_1, uy_1x_2 uy_2x_1, uy_2x_2		$ux_1^2y_1, ux_1^2y_2$ $ux_2^2y_1, ux_2^2y_2$	
0	1	y_1, y_2	x_1, x_2	y_1x_1, y_1x_2 y_2x_2	x_1^2, x_2^2 x_1x_2	$x_1^2y_1, x_1^2y_2$ $x_2^2y_1, x_2^2y_2$	x_1^3, x_2^3 $x_1^2x_2, x_1x_2^2$
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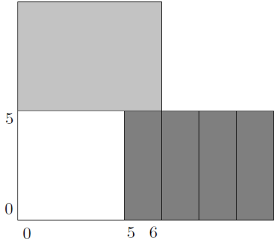

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Cohomology of J_2 over $\mathbb{F}_3$

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	0	1	2	3	4	5	6	7	8
11									
10									
9			$w_{2,9}$						
8								$t^4 w_{7,0}$	
7								$ut^3 w_{7,0}$	
6				$w_{3,6}$	$w_{4,6}$			$t^3 w^*_{7,0}$	$w_{8,6}$
5									
4									
3									
2				$w_{3,2}$				$t w^*_{7,0}$	
1			$w_{2,1}$	$w_{3,1}$ $w^*_{3,1}$				$u w^*_{7,0}$	
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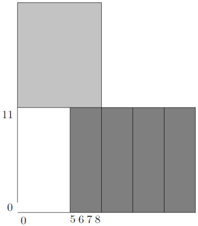

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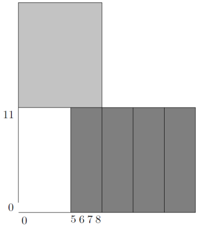

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$$w_{2,1} = uy_1y_2,\ w_{3,1} = uy_1x_1 + uy_2x_2,\ w^*_{3,1} = uy_1x_2 - uy_2x_1,$$

$$w_{3,2} = ty_1x_1 + ty_2x_2,\ w_{3,6} = t^3(y_1x_1 + y_2x_2),\ w_{4,6} = t^3(x_1^2 + x_2^2)$$

Cohomology of J_2 over $\mathbb{F}_3$

Theorem (Leary, 1992)

The ring $H^*(3^{1+2}_+; \mathbb{F}_3)$ is generated by elements y_1, y_1', x_2, x_2', Y_2, Y_2', X_3, X_3', z_6 subject to 22 relations.

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Theorem (D-Garaialde, 2014)

The ring $H^*(J_2;\mathbb{F}_3)$ is the subring of $H^*(3^{1+2}_+;\mathbb{F}_3)$ generated by elements $a_3, b_4, c_4, d_5, e_9, f_{10}, g_{11}, h_{12}$ with

$$a = Yy' - xy - x'y', b = Yx - Y'x', c = x^2 + x'^2 + xY' + x'Y$$

$$d = Xx - X'x', e = z(yx + y'x'), f = z(x^2 + x'^2),$$

$$g = -z(Xx - X'x' + YX'), h = z^2.$$

Cohomology of J_2 over $\mathbb{F}_3$

Theorem (D-Garaialde, 2014)

The ring $H^*(J_2; \mathbb{F}_3)$ is the free graded-commutative algebra on the generators a, b, c, d, e, f, g, h with

$$\deg(a) = 3, \deg(b) = \deg(c) = 4, \deg(d) = 5,$$

$$\deg(e) = 9, \deg(f) = 10, \deg(g) = 11, \deg(h) = 12$$

subject to the following relations:

$$ab = 0, b^2 = 0, bc + ad = 0, bd = 0$$

$$ae = 0, be = 0, fa + ce = 0, bf = ag$$

$$ed = ag, bg = 0, fd + cg = 0, dg = 0$$

$$ef + ach = 0, eg = had, f^2 = c^2h, fg + chd = 0.$$

Cohomology of J_2 over $\mathbb{F}_3$

Theorem (D-Garaialde, 2014)

The ring $H^*(J_2;\mathbb{F}_3)$ is Cohen-Macaulay as it is free and finitely generated over the polynomial subalgebra $\mathbb{F}_3[c_4,h_{12}]$:

$$H^*(J_2;\mathbb{F}_3)=\mathbb{F}_3[c_4,h_{12}]\{1,a_3,b_4,d_5,e_9,f_{10},g_{11},g_{11}a_3\}.$$

Its Poincaré series is

$$P(t)=\frac{1+t^3+t^4+t^5+t^9+t^{10}+t^{11}+t^{14}}{(1-t^4)(1-t^{12})}.$$

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- Fiber $F \to BJ_2 \to BG_2$ is the quotient manifold $G_2/(3^{1+2}_+ \rtimes C_8)$ of dimension 14 with Poincaré duality cohomology.

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Fusion system	Sylow	$\mathrm{Out}_{\mathcal{F}}(B)$	V_0	V_1	V_{-1}	E_0	E_1	E_{-1}
$\mathcal{F}(3^{2k}, 1)$	$B(2k)$	$\mathbb{Z}_2$	$SL_2(3)$	-	-	-	-	-
$\mathcal{F}(3^{2k}, 2)$	$B(2k)$	$\mathbb{Z}_2$	-	$SL_2(3)$	$SL_2(3)$	-	-	-
$3.\mathcal{F}(3^{2k}, 1)$	$B(2k+1)$	$\mathbb{Z}_2$	-	-	-	$SL_2(3)$	-	-
$3.\mathcal{F}(3^{2k}, 2)$	$B(2k+1)$	$\mathbb{Z}_2$	-	-	-	-	$SL_2(3)$	$SL_2(3)$
$3.\mathcal{F}(3^{2k}, 1).2$	$B(2k+1)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	-	-	-	$GL_2(3)$	-	-
$3.\mathcal{F}(3^{2k}, 2).2$	$B(2k+1)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	-	-	-	-	$SL_2(3)$	

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$k \geq 2,\ Z(B(n)) \subseteq V_i \subseteq E_i,\ i = -1, 0, 1.$

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Fusion system	Sylow	$\mathrm{Out}_{\mathcal{F}}(B)$	V_0	V_1	V_{-1}	E_0	E_1	E_{-1}
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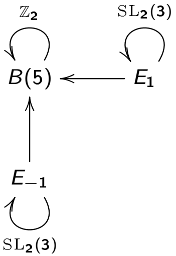

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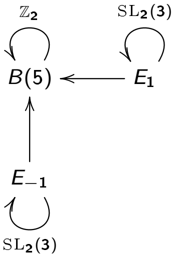

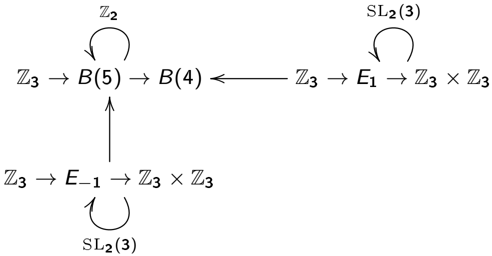

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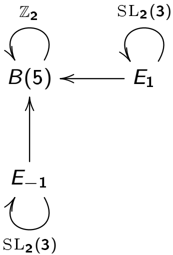

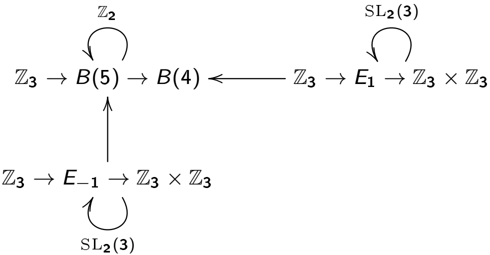

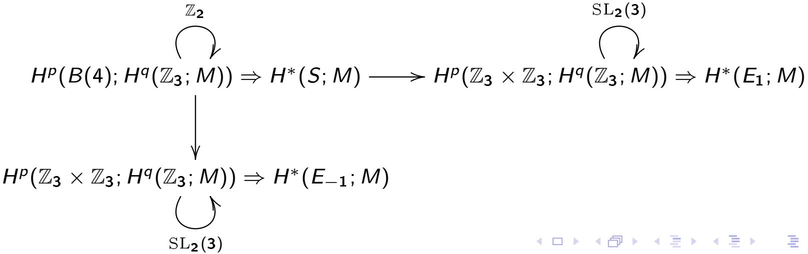

Relation to Carlson's conjecture.

The groups $B(n)$ have order 3^n and maximal nilpotency class $(c = n - 1)$, and fit into split extensions

$$C_{3^k} \times C_{3^k} \to B(2k+1) \to C_3 \text{ or } C_{3^k} \times C_{3^{k-1}} \to B(2k) \to C_3,$$

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Theorem (D-Garaialde-Gonzalez, 2015)

The groups $B(n)$ have isomorphic cohomology rings for $n \geq 3$.

The novelty is that we do not have to compute these rings to prove they are isomorphic. (Although in this case the cohomology ring of $B(3) = 3^{1+2}_+$ is known).
The situation is similar to that of dihedral groups $D_{2^n} = C_{2^{n-1}} \rtimes C_2$, which also have isomorphic cohomology rings for $n \geq 3$.

Relation to Carlson's conjecture.

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Conjecture (Carlson,2005)

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Here, if $|G| = p^n$, then the coclass of G is $cc(G) = n - c(G)$, where $c(G)$ is the nilpotency class of G.

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Relation to Carlson's conjecture.

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Conjecture (D-Garaialde-Gonzalez,2015)

Our (current) proof contains no fatal mistakes.

Thanks!