Cohomology and Quillen categories of finite p-groups and coclass theory

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Introduction

Cohomology groups

Definition

G a finite p-group and $\mathbb{F}$ the field with p elements.

(1) $H^n(G, \mathbb{F})$ is the nth cohomology group of G with coefficients in $\mathbb{F}$ for $n \in \mathbb{N}_0$.

(2) $H^n(G, \mathbb{F})$ is a finite-dimensional $\mathbb{F}$-vectorspace for $n \in \mathbb{N}_0$.

(3) (a) $H^0(G, \mathbb{F}) \cong \mathbb{F}$.

(b) $H^1(G, \mathbb{F}) \cong Hom(G, \mathbb{F})$.

(c) $H^2(G, \mathbb{F}) \leftrightarrow$ equivalence classes of central extensions of G.

Cohomology Rings

Cohomology Rings

(1) $H^*(G, \mathbb{F}) = \oplus_{n \in \mathbb{N}_0} H^n(G, \mathbb{F})$ is the mod-p cohomology ring of G.

(2) $H^*(G, \mathbb{F})$ is a graded $\mathbb{F}$-algebra via the cup-product $H^n(G, \mathbb{F}) \otimes H^m(G, \mathbb{F}) \to H^{n+m}(G, \mathbb{F})$.

(3) $H^*(G, \mathbb{F})$ is infinite-dimensional as $\mathbb{F}$-vectorspace and finitely presented as $\mathbb{F}$-algebra.

(4) $H^*(G, \mathbb{F}) = \mathbb{F} \oplus I$ with $I = \oplus_{n \in \mathbb{N}} H^n(G, \mathbb{F})$ and I is residually nilpotent.

Examples

Dihedral groups

Let D_{2^i} be the dihedral group of order 2^i. Then $H^*(D_{2^i}, \mathbb{F})$ is a commutative graded untial $\mathbb{F}$-algebra with

$$H^*(D_{2^i}, \mathbb{F}) \cong \langle a, b, x, y \mid a^2 = x, b^2 = ab \rangle$$

for $deg(a) = deg(b) = 1$ and $deg(x) = deg(y) = 2$.

Question

Questions

When is $H^*(G,\mathbb{F}) \cong H^*(H,\mathbb{F})$ for $G \ncong H$?

Aim

Investigate isomorphisms among cohomology rings of p-groups.

Groups by Order

Groups by Order

2-groups of small order

2-groups of small order

	number	determined by
2^1	1	
2^2	2	
2^3	5	
2^4	14	Hölder 1893
2^5	51	Miller 1898
2^6	267	Hall & Senior 1964
2^7	2328	James, Newman & O'Brien 1990
2^8	56 092	O'Brien 1991
2^9	10 494 213	Eick & O'Brien 2000
2^{10}	49 487 365 422	Eick & O'Brien 2000

Cohomology rings of small 2-groups

Algorithms

(1) Carlson: algorithm to compute a finite presentation for $H^*(G, \mathbb{F})$. Determined $H^*(G, \mathbb{F})$ for all G with $|G| \mid 64$.

(2) Green & King: improved this method. Determined $H^*(G, \mathbb{F})$ for all G with $|G| \mid 128$ and some larger 2-groups.

Isomorphism testing

King & Eick

Developed an algorithm to test if two mod-p cohomology rings of finite p-groups are isomorphic (as graded algebras).

Isomorphism testing: Step 1

Step 1

Distinguish non-isomorphic rings by invariants

(1) Use the Poincare-Series $\sum_{n \in \mathbb{N}} dim(H^n(G, \mathbb{F}))t^n$.

(2) Use nilpotent quotients A/A^i of the residually nilpotent ideal $A = \oplus_{n \in \mathbb{N}} H^n(G, \mathbb{F})$.

Isomorphism testing: Step 2

Step 2

Developed an algorithm to determine a graded isomorphism between two mod-p cohomology rings if it exists:

(1) Rings are given by finite presentations with homogeneous generators.

(2) There are only finitely many options for the image of each homogeneous generator.

(3) For each collection of possible images use the given presentation to check if this is an homomorphism.

(4) Make this efficient by reducing the number of possible images to a minimum!

Application

Application

Determined isomorphisms types of rings $H^*(G, \mathbb{F})$ for $|G| \mid 64$:

(1) There are 340 isomorphism types of groups.

(2) number of t-tupels of isomorphic rings

t	2	3	4	6	7	9
#	23	11	4	1	1	1

(3) Thus there are 264 isomorphism types of cohomology rings.

Coclass

Coclass

Coclass

Definition (Leedham-Green & Newman)

Let $|G| = p^n$ and $c = cl(G)$. Then $cc(G) = n - c$ is the coclass of G.

Coclass Graphs

Coclass Graphs

Visualize the p-groups of fixed coclass r in a graph $\mathcal{G}(p, r)$:

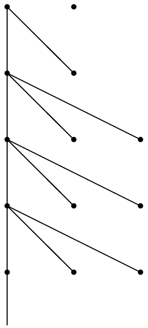
(1) Vertices correspond one-to-one to the isomorphism types of p-groups of coclass r.

(2) $H \to G$ if $G/\gamma(G) \cong H$, where $\gamma(G)$ is the last non-trivial term of the lcs of G.

(3) $\mathcal{G}(p, r)$ is an infinite forest for each p and r.

An example

Example: $\mathcal{G}(2,1)$:



Isomorphisms

Theorem (Carlson)

Let $r \in \mathbb{N}$. Then there are only finitely isomorphism types of rings $H^*(G, \mathbb{F})$ for the infinitely many groups G in $\mathcal{G}(2, r)$.

Questions

Question1

Which groups in $\mathcal{G}(2, r)$ lead to isomorphic cohomology rings?

Question2

What happens in the odd prime case?

The structure of coclass graphs I

Coclass trees

A subtree $\mathcal{T}$ of $\mathcal{G}(p,r)$ is a *coclass tree* if

(1) $\mathcal{T}$ is a descendant tree: it contains all descendants of its root.

(2) $\mathcal{T}$ has a unique infinite path starting at its root (the main line).

(3) $\mathcal{T}$ is maximal with these properties in $\mathcal{G}(p,r)$.

Theorem (Leedham-Green, Shalev)

$\mathcal{G}(p,r)$ consists of finitely many coclass trees and finitely many other vertices.

Pruning

Pruned coclass trees

Let $\mathcal{T}$ be a coclass tree in $\mathcal{G}(p, r)$ with main line $G_0, G_1, \ldots$.
Let $k, l \in \mathbb{N}$ and define subtrees:

(1) $\mathcal{T}_k$ consists of all vertices of distance at most k from the main line.

(2) $\mathcal{T}^{(l)}$ consists of all descendants of G_l in $\mathcal{T}$.

(3) $\mathcal{T}_k^{(l)}$ consists of all descendants of G_l in $\mathcal{T}_k$.

Theorem (Leedham-Green & McKay)

If $p = 2$, then there exists $k = k(p, r)$ with $\mathcal{T}_k = \mathcal{T}$.

Periodicity

Theorem (du Sautoy, Eick & Leedham-Green)

Let $\mathcal{T}$ be a coclass tree in $\mathcal{G}(p, r)$ and let $k \in \mathbb{N}$. Then $\mathcal{T}_k$ is virtually periodic; that is there exists l and d with $\mathcal{T}_k^{(l)} \cong \mathcal{T}_k^{(l+d)}$.

Proofs

(1) du Sautoy: Zeta-functions;

(2) Eick & Leedham-Green: An explicit group theoretic construction: the *coclass families* $(G_0, G_1, \ldots)$.

Conjecture

Conjecture

Let $(G_0, G_1, \ldots)$ be a coclass family. Then $H^*(G_i, \mathbb{F}) \cong H^*(G_j, \mathbb{F})$ for each $i, j \geq l$ for some $l \in \mathbb{N}$.

Quillen categories

Definition

Definition

Let G be a finite p-group. The Quillen category $\mathcal{A}_p(G)$:

(1) objects are the elementary abelian subgroups of G,

(2) morphisms are the injective group homomorphisms induced by conjugation with elements of G.

Definition

A category allows the construction of a limit. We denote

$$\overline{H}^*(G, \mathbb{F}) = \lim_{E \in A_p(G)} H^*(E, \mathbb{F}).$$

Quillen's theorem

Definition

Restriction induces a natural homomorphism

$$\rho : H^*(G, \mathbb{F}) \to \overline{H}^*(G, \mathbb{F}).$$

Quillen's theorem

ρ is an *inseparable isogeny*; That is, every homogeneous element of its kernel is nilpotent and for every element s in its range there exists $n \in \mathbb{N}$ with s^{p^n} in its image.

Quillen categories of p-groups

(Theorem (Green & Eick)

Let $(G_0, G_1, \ldots)$ be a coclass family and let $i, j \geq l$.

(1) $\mathcal{A}_p(G_i)$ is equivalent to $\mathcal{A}_p(G_j)$.

(2) $\overline{H}^*(G_i, \mathbb{F}) \cong \overline{H}^*(G_j, \mathbb{F})$.

Comments

Comments

Let $(G_0, G_1, \ldots)$ a coclass family in $\mathcal{G}(p, r)$ and let $i, j \geq l$.

(1) The isomorphism $\overline{H}^*(G_i, \mathbb{F}) \cong \overline{H}^*(G_j, \mathbb{F})$ can be considered as support for the conjecture $H^*(G_i, \mathbb{F}) \cong H^*(G_j, \mathbb{F})$ for $i, j \leq l$.

(2) Works for all primes p.

Proof

General idea

Let $(G_0, G_1, \ldots)$ be a coclass family.

(1) Determine a special skeleton $\overline{\mathcal{A}}_p(G_i)$ in $\mathcal{A}_p(G_i)$ for $i \in \mathbb{N}$.

(2) Define a functor $F_i : \overline{\mathcal{A}}_p(G_i) \to \overline{\mathcal{A}}_p(G_{i+1})$ for $i \in \mathbb{N}$.

(3) Prove that F induces an equivalence of categories.

Proof

Side-products

Let S be the infinite pro-p-group associated with $(G_0, G_1, \ldots)$.

(1) Determine a special (and finite) skeleton $\overline{\mathcal{A}}_p(S)$.

(2) Define a functor $\hat{F}_i : \overline{\mathcal{A}}_p(G_i) \to \overline{\mathcal{A}}_p(S)$.

(3) $\hat{F}_i$ is not necessarily injective or surjective, but it exhibits how these categories are linked to each other.

Infinite pro-p-groups

Infinite pro-p-groups

Let S be an infinite pro-p-group of finite coclass r.

(1) There exists $T \trianglelefteq S$ with $R = S/T$ finite and $T \cong \mathbb{Z}^d$.

(2) For $U \leq R$ let $\overline{U}$ denote the full preimage of U in S.

The Quillen category for S

Objects in $\mathcal{A}_p(S)$

(1) Let $\mathcal{L} = \{L \leq R \mid L$ elementary abelian and $\overline{L}$ splits over $T\}$.

(2) Let $C_L = \{t(l) \mid l \in L\}$ a complement to T in $\overline{L}$ (for $L \in \mathcal{L}$).

(3) Let $C_L(\delta) = \{t(l)\delta(l) \mid l \in \mathcal{L}\}$ (for $L \in \mathcal{L}$ and $\delta \in Z^1(L, T)$).

(4) Objects of $\mathcal{A}_p(S)$ are $\{C_L(\delta) \mid L \in \mathcal{L}, \delta \in Z^1(L, T)\}$.

The skeleton for S

Objects in $\overline{\mathcal{A}}_p(S)$

(1) Let $T^1(L,T)$ be a transversal to $B^1(L,T)$ in $Z^1(L,T)$.

(2) Objects of $\overline{\mathcal{A}}_p(S)$ are $\{C_L(\delta) \mid L \in \mathcal{L}, \delta \in T^1(L,T)\}$.

(3) Each object in $\mathcal{A}_p(S)$ is conjugate to an object in $\overline{\mathcal{A}}_p(S)$. Hence $\overline{\mathcal{A}}_p(S)$ is a (semi-)skeleton for $\mathcal{A}_p(S)$.

(4) The number of objects in $\overline{\mathcal{A}}_p(S)$ is finite, as $|T^1(L,T)| = |H^1(L,T)|$ is finite and $\mathcal{L}$ is finite.

Finite p-groups of fixed coclass

Coclass families

(1) Let S be an extension of T by R via $\epsilon \in Z^2(R, T)$.

(2) Let $(G_0, G_1, \dots)$ be an associated coclass family.

(3) Then each G_i is an extension of $M_i = T/p^{e+i}T$ by R via $\gamma_i \in Z^2(R, M_i)$.

(4) $H^2(R, M_i) \cong H^2(R, T) \oplus p^i H^3(R, T)$ and $[\gamma_i] = [\epsilon] + p^i \eta$ for some $\eta \in H^3(R, T)$.

Quillen category of G_i

Objects in $\mathcal{A}_p(G_i)$

(1) Let $\mathcal{L}_\eta = \{L \in \mathcal{L} \mid \eta_L = 0\}$.

(2) Let $C_{L,i} = \{t_i(l) \mid l \in L\}$ a complement to T in $\overline{L}$ (for $L \in L_\eta$).

(3) Let $C_{L,i}(\delta) = \{t_i(l)\delta(l) \mid l \in L\}$ (for $L \in \mathcal{L}_\eta$ and $\delta \in Z^1(L, M_i)$).

(4) Objects of $\mathcal{A}_p(G_i)$ are
$\{C_{L,i}(\delta) \times O \mid L \in \mathcal{L}_\eta, \delta \in Z^1(L, M_i), O \leq \Omega(M_i)\}$.

(5) Objects of $\mathcal{A}_p(G_i)$ are induced from objects in $\mathcal{A}_p(S)$.

The skeleton for G_i

Objects in $\overline{\mathcal{A}}_p(G_i)$

(1) Let $T^1(L, M_i)$ be a transversal to $B^1(L, M_i)$ in $Z^1(L, M_i)$.

(2) Objects of $\overline{\mathcal{A}}_p(G_i)$ are $\{C_{L,i}(\delta) \times O \mid L \in \mathcal{L}_\eta, \delta \in T^1(L, M_i), O \leq \Omega(M_i)\}$.

(3) Each object in $\mathcal{A}_p(G_i)$ is conjugate to an object in $\overline{\mathcal{A}}_p(G_i)$. Hence $\overline{\mathcal{A}}_p(G_i)$ is a (semi-)skeleton for $\mathcal{A}_p(G_i)$.

(4) $H^1(L, M_i) = H^1(L, T) \oplus H^2(L, p^{e+i}T)$ implies that $|T^1(L, M_i)| = |H^1(L, T)||H^2(L, T)|$ is independent of i.

Summary

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(1) Determined all isomorphisms between mod-2 cohomology rings of the groups of order dividing 64. (Many of them arise from coclass families.)

(2) If $(G_0, G_1, \ldots)$ is a coclass family in $\mathcal{G}(p, r)$ for arbitrary prime p and $r \in \mathbb{N}$, then $\overline{H}^*(G_i, \mathbb{F}) \cong \overline{H}^*(G_j, \mathbb{F})$ for all $i, j \geq l$.

Conjectures

Conjecture 1

Let p be an arbitrary prime and let $r \in \mathbb{N}$. Then there are only finitely many isomorphism types of mod-p cohomology rings $H^*(G, \mathbb{F}_p)$ for G in $\mathcal{G}(p, r)$.

Conjecture 2

Let p be an arbitrary prime, let $r \in \mathbb{N}$ and let $(G_0, G_1, \ldots)$ be a coclass family in $\mathcal{G}(p, r)$. Then $H^*(G_i, \mathbb{Z})$ can be described by a single parametrised presentation for almost all $i \in \mathbb{N}_0$.