Vector Fields and Group Cohomology

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1: $H_n(SL_2(\mathbb{Z}[\frac{1}{46}]), \mathbb{Z}) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4 \quad (n = 2k + 1 \geq 5)$

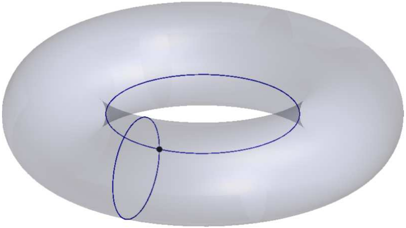
2: $H_5(PSL_4(\mathbb{Z}), \mathbb{Z}) \cong (\mathbb{Z}_2)^{13}$

3: $H^*(\mathrm{SpaceGroup}(4, 32), \mathbb{Z}) \cong$
$$\mathbb{Z}[x, y, z, w]/\langle x^2, y^2, z^2, xzw, yzw, 2(xy + w), zw^2\rangle$$

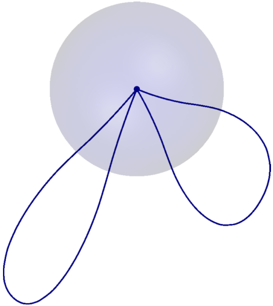
where $H^n(G, \mathbb{Z}) = H^n(BG, \mathbb{Z})$

Bui Anh Tuan, Achill Schürmann, Mathieu Dutour Sikiric

CW-spaces

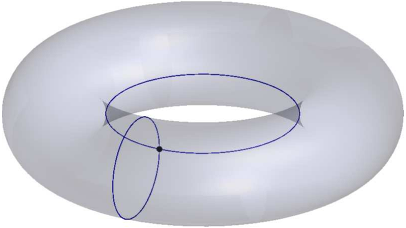


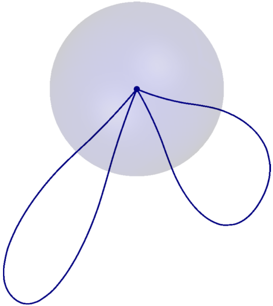
$X = e^0 \cup e^1_x \cup e^1_y \cup e^2$



$Y = e^0 \cup e^1_x \cup e^1_y \cup e^2$

CW-spaces





$X = e^0 \cup e^1_x \cup e^1_y \cup e^2$

$Y = e^0 \cup e^1_x \cup e^1_y \cup e^2$

$$C_*X : \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

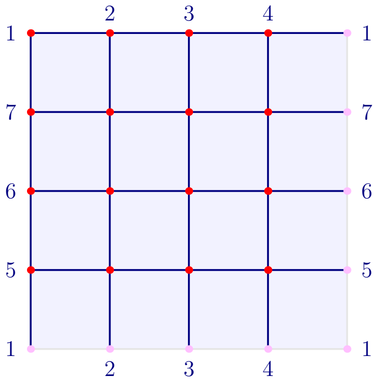
$$C_*Y : \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$\pi_1 X = \langle x, y \ : \ xyx^{-1}y^{-1} = 1 \rangle$$

$$\pi_1 Y = \langle x, y \ : \ 1 = 1 \rangle$$

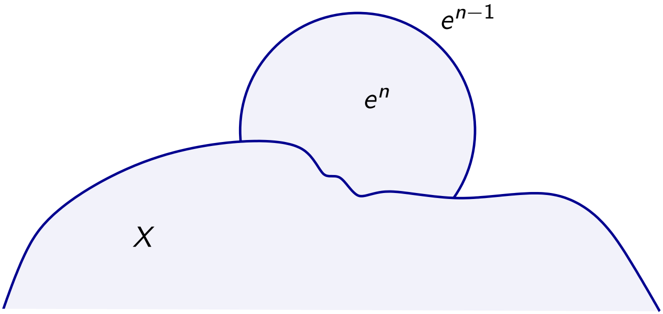
But how can we store higher dimensional CW-spaces on a computer?

Regular CW-spaces



$$C_*X : \cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^{16} \xrightarrow{d_2} \mathbb{Z}^{32} \xrightarrow{d_1} \mathbb{Z}^{16}$$

Simple Homotopy Collapse: $X \cup e^n \cup e^{n-1} \searrow X$



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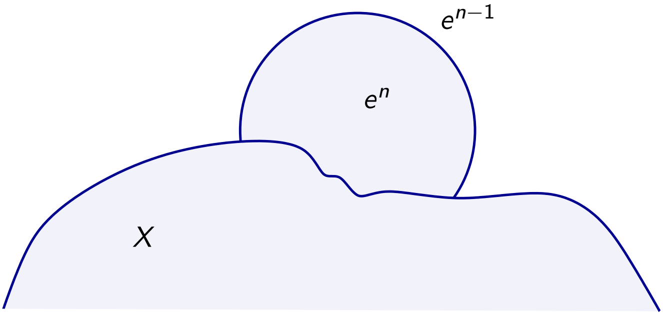
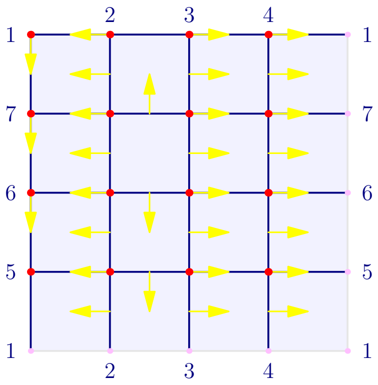


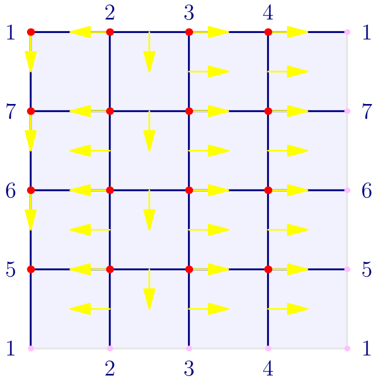
Store in computer as a pair (e^{n-1}, e^n) or arrow $e^{n-1} \longrightarrow e^n$



A discrete vector field is a collection of arrows $e^{n-1} \longrightarrow e^n$ with e^{n-1} in the boundary of e^n and with any cell involved in at most one arrow. It is admissible if there is no chain

$$\cdots (e_1^{n-1}, e_1^n), (e_2^{n-1}, e_2^n), (e_3^{n-1}, e_3^n), \cdots$$

with each e_{i+1}^{n-1} in the boundary of e_i^n and with infinitely many (not necessarily distinct) terms to the right.



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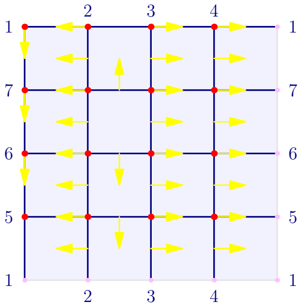
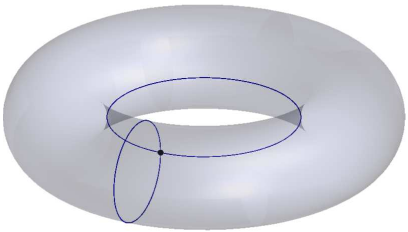
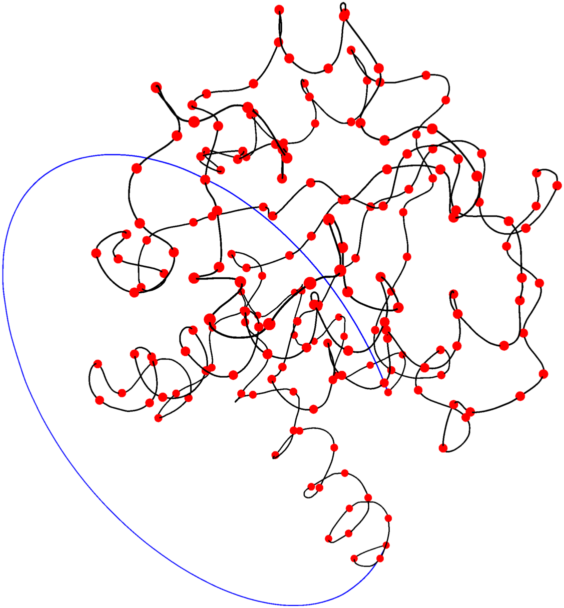
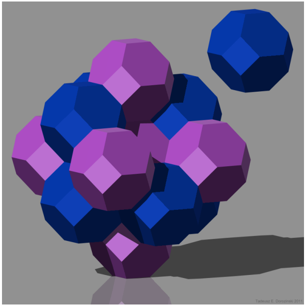
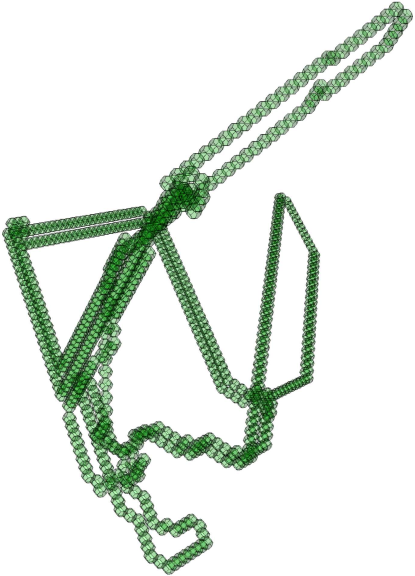


corresponds to







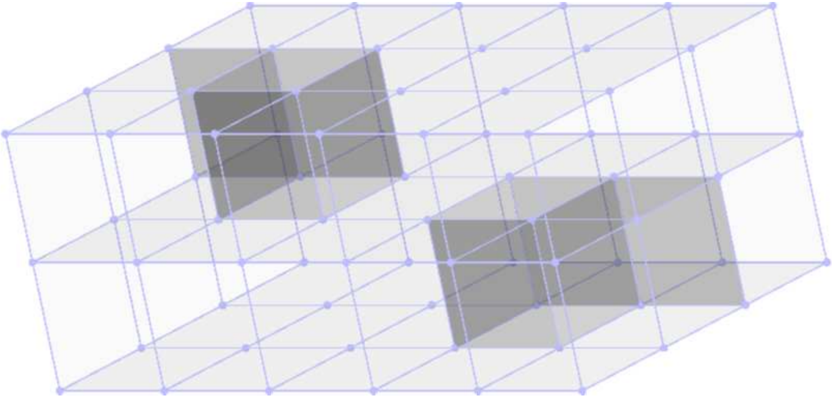


Proposition

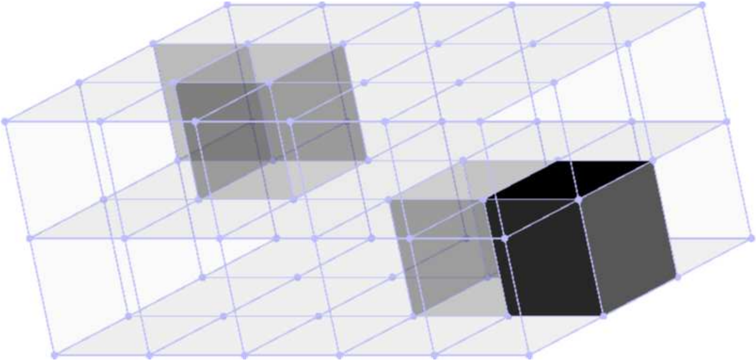
The alpha carbon atoms of a Thermus Thermophilus protein determine a knot K with peripheral system

$$\begin{array}{rcl} \pi_1(\partial K) \cong \langle a,b \mid aba^{-1}b^{-1}\rangle & \to & \pi_1(\mathbb{R}^3 \setminus K) \cong \langle x,y \mid xyx = yxy\rangle \\ a & \mapsto & x^{-2}yx^2y \\ b & \mapsto & x \end{array} .$$

```
gap> K:=ReadPDBfile("1V2X.pdb");
Pure permutahedral complex of dimension 3

gap> Y:=RegularCWComplex(PureComplexComplement(K));;
Regular CW-complex of dimension 3

gap> i:=Boundary(Y);
Map of regular CW-complexes

gap> phi:=FundamentalGroup(i,22495);
[ f1, f2 ] -> [ f1^-3*f2*f1^2*f2*f1, f1 ]
```



The deformation retraction $\mathrm{Bing's\ house} \simeq \{*\}$ is not representable as a discrete vector field.

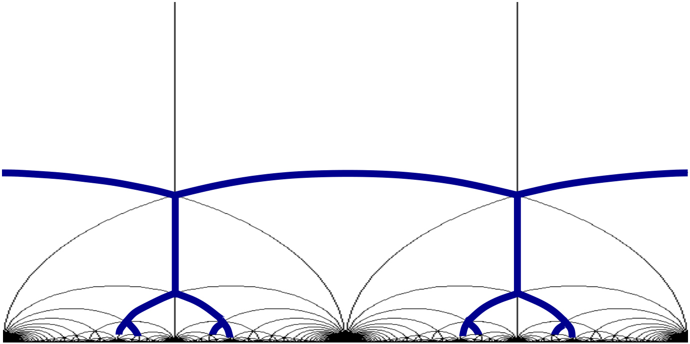


```
gap> B:=BingsModifiedHouse;
Regular CW-complex of dimension 3

gap> Size(ContractedComplex(B));
1
```

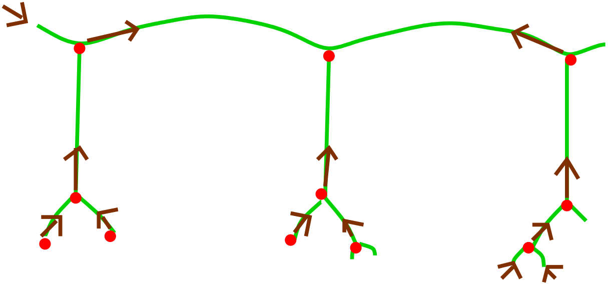
$SL_2(\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

and acts on the cubic tree $\mathcal{T}$ (every vertex of degree 3). Note $S^4 = (ST)^6 = 1$.



Vertices of $\mathcal{T}$ are the left cosets of $\mathcal{U} = \langle ST \rangle$.
Two vertices $x\mathcal{U}$, $y\mathcal{U}$ are connected by an edge iff $x^{-1}y \in \mathcal{U}S\mathcal{U}$.



A discrete vector field on $\mathcal{T}$ with one critical cell is essentially an algorithm for expressing an element $A \in SL_2(\mathbb{Z})$ as a word in S and T.

(K. Conrad) To express

$$A = \begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix} \text{ in terms of } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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note that $17 = 2 \cdot 7 + 3$ and so

$$T^{-2}A = \begin{pmatrix} 3 & 5 \\ 7 & 12 \end{pmatrix}$$

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$$\cdots$$

$$ST^2ST^2ST^3ST^{-2}A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$S^3T^2ST^2ST^3ST^{-2}A = I$$

A CW-resolution of a group G is a regular CW-space $\mathcal{T}$ with

- discrete vector field having precisely one critical cell,
- action of G that permutes the cells, there being finitely many orbits of cells in each dimension.

It is free if every cell stabilizer is free.

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Theorem (Homotopical Perturbation)

A CW-resolution $\mathcal{T}$ of G together with free CW-resolutions $\mathcal{F}_{e^n}$ of the stabilizer G_{e^n} of each orbit representative e^n algorithmically determines a free CW-resolution $\mathcal{F}$ of G.

$\mathcal{F}$ has $(m+n)$-dimensional cells

$$e^n \tilde{\otimes} f^m$$

$e^n \subset \mathcal{T}$, $f^m \subset \mathcal{F}_{e^n}$ and diagonal action

$$g \cdot (e^n \tilde{\otimes} f^m) = (g \cdot e^n) \tilde{\otimes} (g \cdot f^m).$$

For $G = SL_2(\mathbb{Z})$ the free $\mathbb{Z}G$-resolution $C_*\mathcal{F}$ has

$$C_n\mathcal{F} = \mathbb{Z}G \oplus \mathbb{Z}G$$

for $n \geq 1$.

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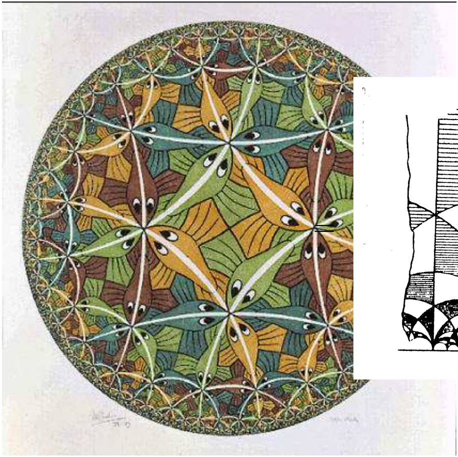
Theorem (Serre)

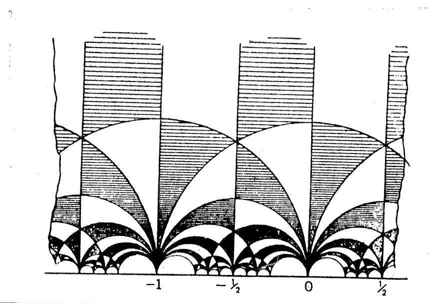
For integer $m > 0$ and prime $gcd(p, m) = 1$

$$SL_2(\mathbb{Z}[1/pm]) \cong SL_2(\mathbb{Z}[1/m]) *_{\Gamma_0(p)} SL_2(\mathbb{Z}[1/m]).$$

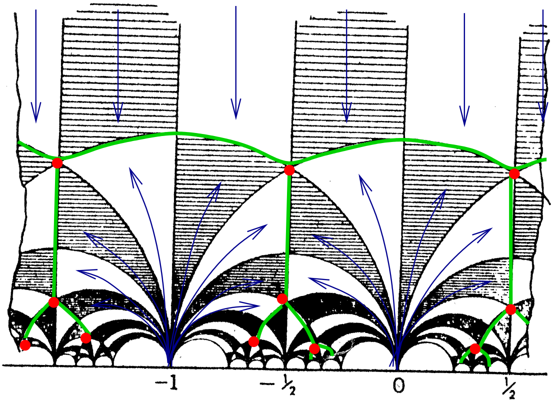
That is, $SL_2(\mathbb{Z}[1/pm])$ acts non-freely on a tree with two orbits of vertices and one orbit of edges. Vertex stabilizers are isomorphic to $SL_2(\mathbb{Z}[1/m])$ and the edge stabilizer $\Gamma_0(p) \leq SL_2(\mathbb{Z}[1/m])$ has finite index.

Voronoi: $PSL_n(\mathbb{Z})$ acts, with finite stabilizers, on a subspace $\mathcal{T}$ of the cone of real $n \times n$ symmetric and positive definite matrices.

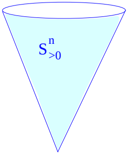




Avner Ash: there is a $PSL_n(\mathbb{Z})$-invariant $\binom{n}{2}$-dimensional homotopy retract $S^n_{wr} \subset S^n_{=1}$

$S^2_{wr} =$ 

A real $n \times n$ symmetric matrix Q is positive definite if $v^t Q v > 0$ for all $v \in \mathbb{R}^2 \setminus \{0\}$. Let $S^n_{>0}$ denote the space of all such matrices.



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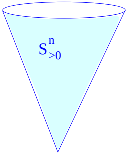

$Q \in S^n_{>0}$, $v \in \mathbb{Z}^n$:	*EXAMPLE* $Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$
$Q[v] = v^t Q v$	$Q[x,y] = x^2 + xy + y^2$
$\min(Q) = \min_{0 \neq v \in \mathbb{Z}^n} Q[v]$	$\min(Q) = 1$
$\mathrm{Min}(Q) = \{v \in \mathbb{Z}^n \ : \ Q[v] = \min(Q)\}$	six vectors
$\rho(v) = v^t v \in S^n_{>0}$	$\rho(1,1) : x^2 + 2xy + y^2$

$Q \in S^n_{>0}$ is perfect if

$$P[v] = \min(Q) \text{ for all } v \in \mathrm{Min}(Q)$$

implies $P = Q$.

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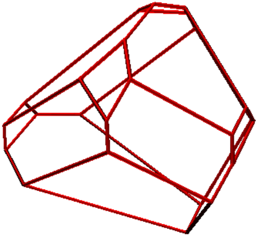
THEOREM (Voronoi)
There are only finitely many perfect forms Q up to $SL_n(\mathbb{Z})$-equivalence, and the cells

$$\mathrm{Dom}(Q) = \left\{ \sum_{v \in \mathrm{Min}(Q)} \lambda_v \rho(v) \ : \ \lambda_v \geq 0 \right\}$$

tessellate (the rational closure of) $S^n_{>0}$.

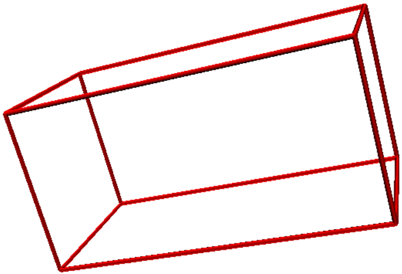
Any n-dimensional crystallographic group G acts on $\mathcal{T} = \mathbb{R}^n$ as affine transformations with finite index translation subgroup $\cong \mathbb{Z}^n$.



$$G = \mathrm{SpaceGroup}(3, 33), v \in \mathbb{R}^3$$

$$D(v) = \{x \in \mathbb{R}^3 \,:\, ||x - v|| \leq ||x - gv|| \text{ for all } g \in G\}$$

Over 50% of 3- and 4-dimensional crystallographic groups have cubical $D(v)$ for suitably chosen v.



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Vector field on a cubical tessellation of $\mathbb{R}^n$ obtained from:

$\mathbb{R}$:

Lemma

Let X, Y be regular CW-spaces with admissible discrete vector fields. There is an admisible discrete vector field on $X \times Y$ with critical cells $e \times e'$ for e critical in X and e' critical in Y. The arrows on $X \times Y$ are

$$s \times f \to t \times f \quad \text{and} \quad e \times s \to e \times t$$

for f an arbitrary cell of Y, e a critical cell in X and $s \to t$ an arbitrary arrow.

For an n-dimensional crystallographic group

$$T \rightarrowtail G \twoheadrightarrow P$$

$$T = \mathbb{Z}^n$$

there is also CW-resolution of G of the form

$$\mathcal{F}_G = \mathcal{F}_T \tilde{\otimes} \mathcal{F}_P$$

with cells

$$e^m \tilde{\otimes} f^n$$

for $e^m \subset \mathcal{F}_T, f^n \subset \mathcal{F}_P$.

For $T \rightarrowtail G \twoheadrightarrow P$ set

$$\gamma_1 T = T, \quad \gamma_{n+1} T = [\gamma_n T, G] \ .$$

Then

$$T/\gamma_n T \rightarrowtail G/\gamma_n T \twoheadrightarrow P$$

yields a resolution

$$\mathcal{F}_{G/\gamma_n T} = \mathcal{F}_{T/\gamma_n T} \tilde{\otimes} \mathcal{F}_P$$

with cells

$$e^m \tilde{\otimes} f^n$$

for $e^m \subset \mathcal{F}_{T/\gamma_n T}, f^n \subset \mathcal{F}_P$.

$G = \mathrm{SpaceGroup}(2, 10)$ has abelian invariants of $T/\gamma_n T$
$...3, 2, 1 = n$

$\ldots [16, 32],\ [16, 16],\ [8, 16],\ [8, 8],\ [4, 8],\ [4, 4],\ [2, 4],\ [2, 2],\ [2],\ [\,]$

for $...3, 2, 1 = n$

$G = \mathrm{SpaceGroup}(2,10)$ has abelian invariants of $T/\gamma_n T$
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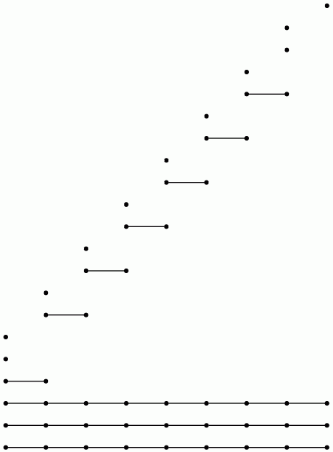
for $...3,2,1 = n$

and $\ldots H_3(G/\gamma_3 T, \mathbb{F}) \longrightarrow H_3(G/\gamma_2 T, \mathbb{F}) \longrightarrow H_3(G/\gamma_1 T, \mathbb{F})$ is



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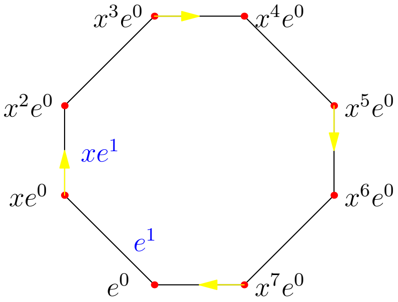

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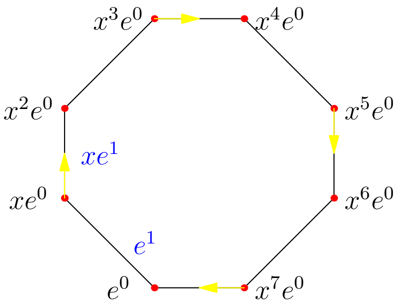

$$\mathcal{F}_{C_8} \xrightarrow{\simeq} \mathcal{F}_{C_4}, \text{ critical cell} \mapsto \text{critical cell}$$

$$\mathcal{F}_{C_8 \times C_8} = \mathcal{F}_{C_8} \times \mathcal{F}_{C_8} \xrightarrow{\simeq} \mathcal{F}_{C_4} \times \mathcal{F}_{C_4} = \mathcal{F}_{C_4 \times C_4}, \text{ critical cell} \mapsto \text{critical cell}$$

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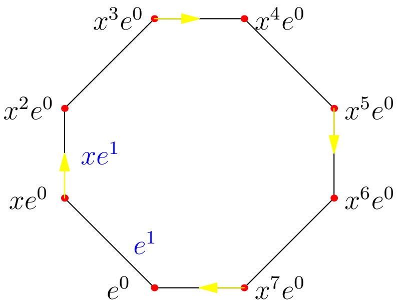

$$\mathcal{F}_{C_8} \xrightarrow{\simeq} \mathcal{F}_{C_4}, \text{ critical cell} \mapsto \text{critical cell}$$

$$\mathcal{F}_{C_8 \times C_8} = \mathcal{F}_{C_8} \times \mathcal{F}_{C_8} \xrightarrow{\simeq} \mathcal{F}_{C_4} \times \mathcal{F}_{C_4} = \mathcal{F}_{C_4 \times C_4}, \text{ critical cell} \mapsto \text{critical cell}$$

$$\mathcal{F}_{G/\gamma_7} = \mathcal{F}_{C_8 \times C_8} \tilde{\otimes} \mathcal{F}_{G/T} \xrightarrow{\simeq} \mathcal{F}_{C_4 \times C_4} \tilde{\otimes} \mathcal{F}_{G/T} = \mathcal{F}_{G/T_5},$$

critical cell $\mapsto$ critical cell

$$\mathcal{F}_{G/\gamma_7} \xrightarrow{\simeq} \mathcal{F}_{G/T_5},$$

$$\text{critical cell} \mapsto \text{critical cell}$$

"induces" a chain map

$$C_*(\mathcal{F}_{G/\gamma_7}) \otimes_{G/\gamma_7 T} \mathbb{F} \longrightarrow \mathcal{F}_{G/\gamma_5 T} \otimes_{G/\gamma_5 T} \mathbb{F}$$

which is a chain isomorphism since

$$H_*(T/\gamma_7, \mathbb{F}) \xrightarrow{\cong} H_*(T/\gamma_5, \mathbb{F}).$$

$$\mathcal{F}_{G/\gamma_7} \xrightarrow{\simeq} \mathcal{F}_{G/T_5},$$

$$\text{critical cell} \mapsto \text{critical cell}$$

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$$H_*(T/\gamma_7, \mathbb{F}) \xrightarrow{\cong} H_*(T/\gamma_5, \mathbb{F}).$$

Proposition

$H_*(G/\gamma_{2k+1} T, \mathbb{F}) \cong H_*(G/\gamma_3 T, \mathbb{F})$ *for* $2k+1 \geq 3$.