Ghost numbers and nilpotency degree for p-group algebras

David J. Green (FSU Jena)
joint with
Fatma Altunbulak Aksu (Çankaya)

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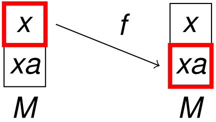
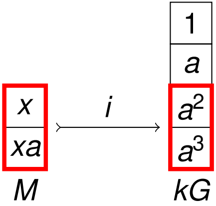
Example: Stable category

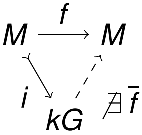
- $kG = \mathbb{F}_2 C_4 \cong \mathbb{F}_2[a]/(a^4)$
- Module M: generator x, relation $xa^2 = 0$.
- $f\colon M \to M$, $x \mapsto xa$.



- Embedding $\begin{array}{rcl} i\colon M & \rightarrowtail & kG \\ x & \mapsto & a^2 \end{array}$

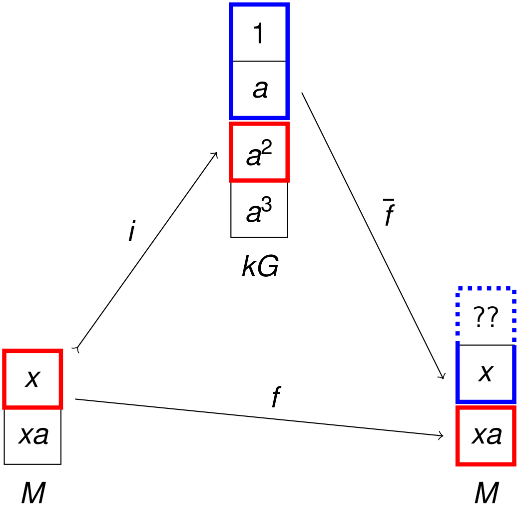


- f doesn't extend along i:



- So $f \neq 0$ in the stable category $\mathrm{stmod}(kG)$.

Why no extension?

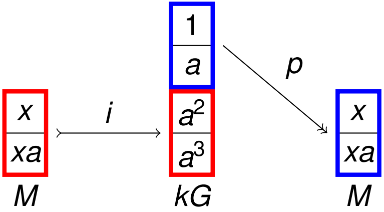


Tate cohomology

Tate cohomology $\hat{H}^r(G, M) := \underline{\mathrm{Hom}}_{kG}(k, \Omega^{-r} M)$.

In our example:

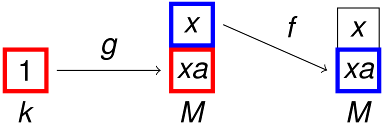
- $kG/\operatorname{Im}(i) \cong M$.



- So $\Omega^{-1}(M) \cong M$. i.e. M periodic, period 1.
- Hence: $\hat{H}^r(G, M) \cong \underline{\mathrm{Hom}}_{kG}(k, M) \cong k$ for all r.

f is a ghost

- Let $g \in \hat{H}^r(G, M)$.
- So g a map $k \to M$.
- Then $\mathrm{Im}(g) \subseteq \ker(f)$:



- Hence $f_* \colon \hat{H}^*(G, M) \to \hat{H}^*(G, M)$ is zero.
- f is a *ghost*: zero in Tate cohomology.
- But not zero in $\mathrm{stmod}(kG)$.
- f is a *non-zero* ghost.

Ghost number

- $M \xrightarrow{f} N$ a *ghost* $:\Leftrightarrow$ $f_* \colon \hat{H}^*(G, M) \to \hat{H}^*(G, N)$ vanishes.
- The ghosts form an ideal in category stmod(kG).
- Chebolu–Christensen–Mináč: *ghost number*

$$\operatorname{gn}(G) := \text{nilpotency degree of ghost ideal}\,.$$

- Example shows $\operatorname{gn}(C_4) \geq 2$.
- In fact $\operatorname{gn}(C_4) = 2$.
- Q: What can one say about $\operatorname{gn}(G)$?
- Today: G a p-group; k has characteristic p.

Ghost number: what we know

Chebolu–Christensen–Mináč (2008):

- $\mathrm{gn}(C_{p^n}) = \begin{cases} 2^{n-1} & p = 2 \\ \frac{p^n-1}{2} & p \text{ odd} \end{cases}.$
- For p-groups: $H \leq G \Rightarrow \mathrm{gn}(H) \leq \mathrm{gn}(G)$.
- $\mathrm{gn}(C_2^n) = n$.

Christensen–Wang (2014):

- $\mathrm{gn}(C_3 \times C_3) = 3$.
- $\mathrm{gn}(D_{2^n}) = 2^{n-2} + 1$.
- $\mathrm{gn}(Q_8) \in \{3, 4\}$.
- **Conjecture** For $|G| = p^n$ have

$$\mathrm{gn}(C_p^n) \leq \mathrm{gn}(G) \leq \mathrm{gn}(C_{p^n}) .$$

Using the nilpotency index one can say more.

Nilpotency index $t(G)$

- Jacobson radical $J(kG)$ is nilpotent.
- G p-group: J is augmentation ideal.
- Nilpotency index $t(G) :=$ smallest r with $J(kG)^r = 0$.
- Example: $kG = \mathbb{F}_2 C_4 = \mathbb{F}_2[a]/(a^4)$.
  $J = (a)$, so $t(C_4) = 4$.
- Example: $kG = \mathbb{F}_2 Q_8$ generated by a, b with relations

  $$a^2 = b^2 \qquad b^4 = 0 \qquad [a, b] \in b^2 + J^3 \,.$$

  Standard monomials $1, a, b, ba, b^2, b^2 a, b^3, b^3 a$.
  $t(Q_8) = 5$.
- General case: Jennings' Theorem computes $t(G)$.

Nilpotency index and ghost number

Chebolu–Christensen–Mináč (2008):

$$\mathrm{gn}(G) \lneq t(G) \leq |G| \; .$$

Christensen–Wang (2014):
If $1 \neq C \trianglelefteq G$ is cyclic and normal then

$$t(G/C) \leq \mathrm{gn}(G) \, .$$

Potential approach to conjecture: show

- G not cyclic $\Rightarrow$ $t(G) \leq \mathrm{gn}(C_{p^n})$; and
- G not elementary abelian $\Rightarrow$ $t(G/C) \geq t(C_p^n)$.

Jennings' Theorem

Jennings series $\Gamma_r(G)$

- $\Gamma_1(G) = G$;
- $\Gamma_r(G) = \left\langle [g, g'], h^p \,\middle|\, \begin{matrix} g \in \Gamma_a, g' \in \Gamma_b & a + b \geq r \\ h \in \Gamma_c & pc \geq r \end{matrix} \right\rangle$

Jennings' Theorem:

- $p^{n_r} := |\Gamma_r : \Gamma_{r+1}|$
- $t(G) = 1 + \sum_r rn_r(p - 1)$.

Examples:

- $G = 3^{1+2}_+$: $\Gamma_1 = G$, $\Gamma_2 = Z(G) \cong C_3$, $\Gamma_3 = 1$
  $n_1 = 2$, $n_2 = 1$, $n_r = 0$ $\quad t(G) = 1 + 2(1 \cdot 2 + 2 \cdot 1) = 9$.
- $G = 3^{1+2}_-$: $\Gamma_1 = G$, $\Gamma_2 = \Gamma_3 = Z(G) \cong C_3$, $\Gamma_4 = 1$
  $n_1 = 2$, $n_2 = 0$, $n_3 = 1$ $\quad t(G) = 1 + 2(1 \cdot 2 + 3 \cdot 1) = 11$.

Koshitani and the upper bound

Shigeo Koshitani (1977): Let $|G| = p^n$

- $t(C_{p^n}) = p^n$
- G not cyclic $\Rightarrow\ t(G) \leq p^{n-1} + p - 1$
- $t(G) = p^{n-1} + p - 1 \ \Rightarrow\ G$ has cyclic subgroup of index p.

Deduce (A^{2}G):

- Christensen-Wang upper bound $\operatorname{gn}(G) \leq \operatorname{gn}(C_{p^n})$.
- Equality $\Rightarrow\ p = 2$ and $G \in \{C_2 \times C_{2^{n-1}}, Q_{2^n}, SD_{2^n}, Mod_{2^n}\}$.
- Chebolu–Christensen–Mináč: Equality for $C_2 \times C_{2^{n-1}}$. Other cases open.

Motose and the lower bound

Kaoru Motose (1978): Let $|G| = p^n$

- $t(C_p^n) = n(p-1)+1$
- G not elementary abelian $\Rightarrow$ $t(G) \geq (n+1)(p-1)+1$.
- $t(G) = (n+1)(p-1)+1 \;\Rightarrow\; |\Phi(G)| = p$.

Deduce (A^{2}G):

- Suppose have $C \trianglelefteq G$ order p, with $C \neq \Phi(G)$. Then

$$\operatorname{gn}(G) \geq t(G/C) \geq t(C_p^n) > \operatorname{gn}(C_p^n)\,.$$

- Remaining cases: $G = C_{p^2}$ or (almost) extraspecial.
- $p=2$: easier, since know $\operatorname{gn}(C_2^n) = n$.
- Christensen–Wang and Motose results actually stronger.
- Hence Christensen–Wang lower bound proved except for
  - $G = p_+^{1+2n}$, p odd.
  - $G = p_-^{1+2}$, $p \in \{3,5\}$. (Both cases still open.)

More about Q_8

- $kG = \mathbb{F}_2 Q_8$ generated by a, b with relations

$$a^2 = b^2 \qquad b^4 = 0 \qquad [a,b] = b^2(1 + a + b + ba)\,.$$

- $(b^2) = [kG, kG] = Z(kG)$.
- $b^2 kG \cong kG/b^2 kG \cong kV_4$
- Hence $M := b^2 kG$ one-periodic.
- Christensen–Wang:
  - $f_a(b^2 x) = b^2 ax$ and $f_b(b^2 x) = b^3 x$ both ghosts;
  - $f_a f_b \neq 0$ in $\mathrm{stmod}(kG)$.
  - That demonstrates $\mathrm{gn}(Q_8) \geq 3$.
- $I \subseteq kG$ any right ideal $\quad \Rightarrow$[1] $\quad \exists\, r$ with

$$J^r(kG) \subseteq I \subseteq J^{r-1}(kG)\,.$$

[1] Thank you to Nadia Mazza and Peter Symonds for pointing out that this fails over $\mathbb{F}_4$.