Saturated fusion systems over a Sylow p-subgroup of $\mathrm{Sp}_4(p^n)$

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Joint work with
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If G is a finite group with Sylow p-subgroup S, the information about fusion in G is basically encoded in the following category:

Definition. Let G be a group and S a subgroup of G. The **fusion category** $\mathcal{F}_S(G)$ is the category whose objects are all subgroups of S and, for all subgroups $P, Q \leq S$,

$$\begin{aligned} \mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) &:= \mathrm{Hom}_G(P, Q) \\ &:= \{c_g \colon P \to Q \mid g \in G \text{ with } P^g \leq Q\}. \end{aligned}$$

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If G is finite and $S \in \mathrm{Syl}_p(G)$ then $\mathcal{F}_S(G)$ has "nice properties".

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Set

$$P^{\mathcal{F}} := \{Q \leq S \colon \mathrm{Iso}_{\mathcal{F}}(P,Q) \neq \emptyset\}.$$

The subgroups of S in $P^{\mathcal{F}}$ are called **$\mathcal{F}$-conjugate** to P.

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(I) $\mathrm{Aut}_S(Q) \in Syl_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.

(II) Any $\mathcal{F}$-morphism with image Q can be extended. More precisely, if $\varphi \in \mathrm{Iso}_{\mathcal{F}}(R, Q)$, then for a certain subgroup $N_\varphi \leq N_S(R)$, there exists $\hat{\varphi} \in \mathrm{Mor}_{\mathcal{F}}(N_\varphi, S)$ with $\hat{\varphi}|_R = \varphi$.

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If $\mathcal{F}$ is saturated, then (I) and (II) hold for any **fully normalized** subgroup Q, i.e. for any subgroup Q such that

$$|N_S(Q)| \geq |N_S(Q^*)| \text{ for any } Q^* \in Q^{\mathcal{F}}.$$

Realizing fusion systems

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Every saturated fusion system can be realized as the fusion category $\mathcal{F}_S(G)$ of a finite group G with p-subgroup S (Park) and as the fusion category $\mathcal{F}_S(G)$ of an infinite group G with Sylow subgroup S (Leary–Stancu, Robinson).

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By work of Broto, Castellana, Grodal, Levi and Oliver every constrained fusion systems is the fusion category $\mathcal{F}_S(G)$ of a finite group G with a Sylow p-subgroup S.

Exotic fusion systems

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There are many other open questions....

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It is (at least theoretically) easy to build fusion systems on S. We can take any collection of injective group homomorphisms between subgroups of S and take the fusion system they generate on S. So the second question reduces to:

(Q2') Given a fusion system $\mathcal{F}$ on S, how can we check that $\mathcal{F}$ is saturated?

Essential subgroups

- Let G be a finite group and $H \leq G$. Then H is called **strongly p-embedded** if $p \mid |H|$ and p does not divide $|H \cap H^g|$ for any $g \in G \backslash H$.

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- Every $\mathcal{F}$-conjugate of an essential subgroup is essential, so we can talk about essential classes.

Control of fusion

Theorem (The Alperin–Goldschmidt fusion theorem)

Let $P_1, \ldots, P_n$ be representatives of the essential classes. Then

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S),\ \mathrm{Aut}_{\mathcal{F}}(P_i) \mid 1 \leq i \leq n \rangle_S.$$

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Note that we can take $P_1, \ldots, P_n$ fully normalized. So we can assume $\mathrm{Aut}_S(P_i) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P_i))$.

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$$\mathcal{F} := \langle H_S,\ H_i \mid 1 \leq i \leq n \rangle.$$

We want to prove that $\mathcal{F}$ is saturated.

Sufficient conditions for saturation

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We use this to give a sufficient condition for saturation of a fusion system generated by certain groups of automorphisms (which look like $\mathcal{F}$-automorphism groups of essential subgroups).

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- Work of Oliver and Craven–Oliver (next talk).

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If $p = 2$ then both radical subgroups are elementary abelian and the only candidates for essential subgroups. Unless $\mathcal{F}$ is constrained, $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $\mathrm{PSp}_4(q) \le G \le \mathrm{Aut}(\mathrm{PSp}_4(q))$ and $S \in \mathrm{Syl}_p(G)$.

Sylow p-subgroups of $\mathrm{Sp}_4(q)$, odd p

Assume now p is odd. Then one radical subgroup is elementary abelian, the other one is isomorphic to a Sylow p-subgroup of $\mathrm{SL}_3(q)$. Let V be the elementary abelian radical subgroup.

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The possible candidates for essential subgroups are the elements of $\{V\} \cup \mathcal{P} \cup \mathcal{E}$.

Sylow p-subgroups of $\mathrm{Sp}_4(q)$, odd p

Assume now p is odd. Then one radical subgroup is elementary abelian, the other one is isomorphic to a Sylow p-subgroup of $\mathrm{SL}_3(q)$. Let V be the elementary abelian radical subgroup.

Write $\mathcal{P}$ for the set of subgroups of S which are isomorphic to a Sylow p-subgroup of $\mathrm{SL}_3(q)$, and $\mathcal{E}$ for the set of elementary abelian subgroups of S of order q^2 which are not contained in V.

The possible candidates for essential subgroups are the elements of $\{V\} \cup \mathcal{P} \cup \mathcal{E}$.

Assume $\mathrm{Aut}_{\mathcal{F}}(V)$ is a $\mathcal{K}$-group.

If $E \in \mathcal{E}$ is essential, then $\mathrm{Aut}_{\mathcal{F}}(E)$ is isomorphic to a subgroup of $\Gamma L_2(q)$ containing $SL_2(q)$ of index prime to p, and E is a natural module.

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If V is essential, then there are two possibilities:

(V1) $\mathrm{Aut}_{\mathcal{F}}(V)$ has a normal subgroup isomorphic to $\mathrm{SL}_2(q)$ and V is the 3-dimensional Lie module for $\mathrm{SL}_2(q)$.

(V2) $q = 9$, $\mathrm{Aut}_{\mathcal{F}}(V)$ is of shape $2.L_3(4) \cdot 2$ or $2.L_3(4) \cdot 2^2$ and acts irreducibly on V.

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(4) $q = p \in \{3, 5\}$, "many" subgroups in $\mathcal{E}$ are essential, at most one subgroup in $\mathcal{P}$ is essential. If V is essential then (V1) holds. (Several cases)

Thank you!!!