Cartan-Bott periodicity for maximal compact subalgebras of real Kac-Moody algebras of type E_n

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Overview

1 Basics

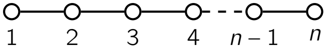
2 Maximal compact subalgebras of split real Kac-Moody algebras

3 Spin representations

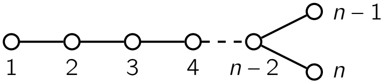
4 Cartan-Bott periodicity

Simply laced Dynkin diagrams

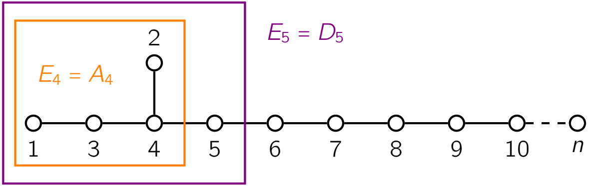
- Simply laced Dynkin Diagram Π $\hat{=}$ undirected simple graph
- rank of Π equals number of vertices of Π
- Examples:
  1. A_n:



  2. D_n:



  3. E_n:



Dynkin diagrams are ubiquitous

- Dynkin diagrams (in particular those of type ADE) appear in ...
  - Lie theory $\rightsquigarrow$ classification of semisimple Lie algebras, Lie groups
  - classification of finite simple groups $\rightsquigarrow$ finite groups of Lie type
  - classification of reflection groups $\rightsquigarrow$ Coxeter diagrams
  - incidence geometry, e.g. building theory
- Kac-Moody groups and algebras are infinite dimensional generalizations of semisimple Lie groups and algebras.
- For this talk: Π simply laced diagram on vertex set $\{1, \ldots, n\}$
- We write $i \sim j$ if vertices i and j are connected

Let's stay real

- Unless stated otherwise, all algebras and groups are over the reals!
- In particular: not over the complex numbers!

Kac-Moody algebras by example

- The split real Kac-Moody algebra of type A_n, $n \geq 1$, is

$$\mathfrak{g}(A_n) := \mathfrak{sl}_{n+1}(\mathbb{R}) := \{a \in M_{n+1}(\mathbb{R}) \mid \mathrm{tr}(a) = 0\}$$

- This is a Lie algebra via $[A, B] := AB - BA$
- Generated (as Lie algebra) by

$$e_1 := \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, f_1 := \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \ldots, e_n := \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix}, f_n := \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}$$

- $\langle e_i, f_i \rangle \cong \mathfrak{sl}_2(\mathbb{R}), \quad \langle e_i, f_i, e_j, f_j \rangle \cong \begin{cases} \mathfrak{sl}_3(\mathbb{R}) & \text{if } |i-j| = 1 \\ \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R}) & \text{if } |i-j| > 1 \end{cases}$
- For simply laced diagram Π, define split real Kac-Moody algebra $\mathfrak{g}(\Pi)$ by gluing together copies of $\mathfrak{sl}_2(\mathbb{R})$ (local-to-global principle)
- $\rightsquigarrow$ presentation by generators and relations

Maximal compact subalgebra

- Split real Kac-Moody algebra of type A_n:

$$\mathfrak{g}(A_n) := \mathfrak{sl}_{n+1}(\mathbb{R}) := \{a \in M_{n+1}(\mathbb{R}) \mid \operatorname{tr}(a) = 0\}$$

- Maximal compact subalgebra:

$$\mathfrak{k}(A_n) := \mathfrak{so}_{n+1}(\mathbb{R}) := \{a \in M_{n+1}(\mathbb{R}) \mid a^t = -a\} \subset \mathfrak{sl}_{n+1}$$

- $\rightsquigarrow$ centralizer of Cartan–Chevalley-involution $\omega : a \mapsto -a^t$
(general case: ω interchanges $e_i \leftrightarrow -f_i$)

- Generated (as Lie algebra) by

$$x_1 := e_1 - f_1 = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ & \begin{smallmatrix} 0 & \\ & \ddots \end{smallmatrix} \end{pmatrix}, \ldots, x_n := e_n - f_n = \begin{pmatrix} \begin{smallmatrix} 0 & \\ & \ddots \end{smallmatrix} & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$$

- Relations: $\operatorname{ad}_{x_i}^2(x_j) = [x_i, [x_i, x_j]] = -x_j$ if $|j - i| = 1$,
$[x_i, x_j] = 0$ otherwise

Berman's theorem

Theorem 1 (Berman 1989)

- Π *simply laced diagram with vertex set* $\{1, \ldots, n\}$
- $\mathfrak{k} := \mathfrak{k}(\Pi)$ *maximal compact subalgebra of* $\mathfrak{g}(\Pi)$

$\Rightarrow \mathfrak{k} \cong$ *real Lie algebra with generators* $X_1, \ldots, X_n$*, relations*

$$\begin{aligned} [X_i, [X_i, X_j]] &= -X_j && \text{if } i \sim j \\ [X_i, X_j] &= 0 && \text{if } i \nsim j \end{aligned}$$

- Berman also considered non-simply-laced Dynkin diagrams
- generating set $X_1, \ldots, X_n$ of $\mathfrak{k}$ as above: Berman generators

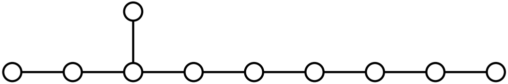
Computing $\mathfrak{k}(\Pi)$ for some small diagrams

n	$\dim(\mathfrak{k}(A_n))$	$\dim(\mathfrak{k}(D_n))$	$\dim(\mathfrak{k}(E_n))$
1	1		
2	3		
3	6	6	
4	10	12	10
5	15	20	20
6	21	30	36
7	28	42	63
8	36	56	120
9	45	72	∞
$\vdots$	$\vdots$	$\vdots$	
n	$\binom{n+1}{2}$	$2\binom{n}{2}$	

- $\mathfrak{k}(A_n) = \mathfrak{so}(n+1)$, $\mathfrak{k}(D_n) = \mathfrak{so}(n) \oplus \mathfrak{so}(n)$
- $\dim(\mathfrak{k}(E_n)) < \infty \Leftrightarrow n \leq 8$

An observation about E_{10}

E_{10}:



- E_{10} and other members of the E_n family play a role string theory.
- Physicists (Damour et al., Henneaux et al.) observed:

  $\mathfrak{k}(E_{10})$ *admits a 32-dimensional complex representation extending the spin representation of its regular* $\mathfrak{so}(10)$ *subalgebra*

- $\rightsquigarrow$ "(generalized) spin representation"
- Induces epimorphism $\mathfrak{k}(E_{10}) \twoheadrightarrow \mathfrak{so}(32)$.
- Surprise (?): $\mathfrak{g}(E_{10})$ is simple, but $\mathfrak{k}(E_{10})$ is not

Spin representations of $\mathfrak{k}$

- Π simply-laced Dynkin diagram
- $\mathfrak{k} := \mathfrak{k}(\Pi)$ with Berman generators $X_1, \ldots, X_n$

Definition 2 (Hainke-Köhl-Levy 2011)

Lie algebra representation $\rho : \mathfrak{k} \to \mathfrak{gl}_s(\mathbb{R})$ is spin-$\frac{1}{2}$ representation if

$$\rho(X_i)^2 = -\frac{1}{4} I_s \quad \text{for } 1 \le i \le n$$

Theorem 3 (Hainke-Köhl-Levy 2011)

- $\mathfrak{k}$ *admits non-trivial spin representation* ρ,
- $\rho(\mathfrak{k})$ *is compact Lie subalgebra of* $\mathfrak{sl}_s(\mathbb{R})$
- $\rho(\mathfrak{k})$ *is semisimple if* Π *contains no isolated nodes*

Spin representations are finite-dimensional

- generators $X_1, \ldots, X_n$, relations $[X_i,[X_i,X_j]] = -X_j$ if $i \sim j$
  $[X_i, X_j] = 0$ if $i \nsim j$
- $\rho : \mathfrak{k} \to \mathfrak{gl}_s(\mathbb{R})$ such that $\rho(X_i)^2 = -\frac{1}{4} I_s$
- For $1 \le i \ne j \le n$, set $A := \rho(X_i)$, $B := \rho(X_j)$ $\rightsquigarrow$ $A^2 = B^2 = -\frac{1}{4} I_s$
- $i \nsim j$: $[A, B] = 0 \Leftrightarrow AB = BA$
- $i \sim j$: $[A,[A,B]] = -B \quad \Leftrightarrow \quad A^2B - 2ABA + BA^2 = -B$

$$\Leftrightarrow \quad -\frac{1}{2}B - 2ABA = -B \quad \Leftrightarrow \quad -2ABA = -\frac{1}{2}B$$

$$\Leftrightarrow \quad -4ABA^2 = -BA \Leftrightarrow \quad AB = -BA$$

- $\rho(\mathfrak{k}) \subset \langle \rho(X_1), \ldots, \rho(X_n) \rangle \Rightarrow \dim \rho(\mathfrak{k}) \le 2^n$

Maximal spin representation

- $A(\Pi)$: associative $\mathbb{R}$-algebra with presentation

$$\left\langle Y_1, \ldots, Y_n \,\middle|\, Y_i^2 = -1, \quad \begin{matrix} Y_i Y_j = -Y_j Y_i & \text{if } i \sim j \\ Y_i Y_j = Y_j Y_i & \text{if } i \nsim j \end{matrix} \right\rangle$$

- $\rho_{max} : \mathfrak{k}(\Pi) \to A(\Pi) : X_i \mapsto \frac{1}{2} A_i$ is maximal spin representation
- Any other spin representation ρ factors through ρ_{max}
- $\dim(\operatorname{im} \rho_{max}) < \infty$, even for $\Pi = E_n$, $n \geq 9$
- Image is non-trivial, so $\mathfrak{k}(E_n)$ is *not* simple for $n \geq 9$
- In fact $\mathfrak{k}(E_n) \cong \ker \rho_{max} \oplus \operatorname{im} \rho_{max}$
- What is $\operatorname{im} \rho_{max}$ for E_n?

Some experiments

- A_n and D_n: nothing new
- For E_n, however...
- Can we explain this?

n	$\dim \rho_{max}(\mathfrak{k}(E_n))$	
4	10	$\binom{4+1}{2}$
5	20	$2\binom{4+1}{2}$
6	36	$\binom{8+1}{2}$
7	63	$8^2 - 1$
8	120	$\binom{16}{2}$
9	240	$2\binom{16}{2}$
10	496	$\binom{32}{2}$
11	1023	$32^2 - 1$
12	2080	$\binom{64+1}{2}$
13	4160	$2\binom{64+1}{2}$
14	8256	$\binom{128+1}{2}$
15	16383	$128^2 - 1$
16	32640	$\binom{256}{2}$

Clifford algebras

- For $n \in \mathbb{N}$, let Cl_n be the associative $\mathbb{R}$-algebra with presentation

$$\left\langle v_1, \ldots, v_n \,\middle|\, v_i^2 = 1,\ v_i v_j = -v_j v_i \right\rangle$$

- Cl_n is a real Clifford algebra
- $\dim \mathrm{Cl}_n = 2^n$, with basis $\{v_1^{e_1} \cdots v_n^{e_n} \mid e_1, \ldots, e_n \in \{0, 1\}\}$
- Clifford algebras are well-known objects with rich theory
- As usual, Cl_n becomes Lie algebra by setting $[A, B] := AB - BA$
- $\mathfrak{k}(\mathrm{Cl}_n) :=$ maximal compact semisimple Lie subalgebra of Cl_n

Cartan-Bott periodicity for Cl_n

Theorem 4 (Cartan 1908; Bott 1960ies)

For $n \geq 2$, the isomorphism types of Cl_n and $\mathfrak{k}(\mathrm{Cl}_n)$ are as follows.

n mod 8	Cl_n	$\mathfrak{k}(\mathrm{Cl}_n)$
0	$\mathfrak{gl}(2^{\frac{n}{2}}, \mathbb{R})$	$\mathfrak{so}(2^{\frac{n}{2}})$
1	$\mathfrak{gl}(2^{\frac{n-1}{2}}, \mathbb{R}) \oplus \mathfrak{gl}(2^{\frac{n-1}{2}}, \mathbb{R})$	$\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}})$
2	$\mathfrak{gl}(2^{\frac{n}{2}}, \mathbb{R})$	$\mathfrak{so}(2^{\frac{n}{2}})$
3	$\mathfrak{gl}(2^{\frac{n-1}{2}}, \mathbb{C})$	$\mathfrak{su}(2^{\frac{n-1}{2}})$
4	$\mathfrak{gl}(2^{\frac{n-2}{2}}, \mathbb{H})$	$\mathfrak{sp}(2^{\frac{n-2}{2}})$
5	$\mathfrak{gl}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathfrak{gl}(2^{\frac{n-3}{2}}, \mathbb{H})$	$\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}})$
6	$\mathfrak{gl}(2^{\frac{n-2}{2}}, \mathbb{H})$	$\mathfrak{sp}(2^{\frac{n-2}{2}})$
7	$\mathfrak{gl}(2^{\frac{n-1}{2}}, \mathbb{C})$	$\mathfrak{su}(2^{\frac{n-1}{2}})$

Observation: $\dim \mathfrak{k}(\mathrm{Cl}_n) = \dim \rho_{max}(\mathfrak{k}(E_n))$

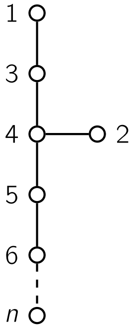
Cartan-Bott periodicity for $\rho_{max}(\mathfrak{k}(E_n))$

Recall: $\rho_{max} : \mathfrak{k}(E_n) \to A(E_n)$ maximal spin representation

Theorem 5 (H.-Köhl 2014)

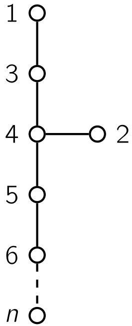
For $n \geq 4$: $\rho_{max}(\mathfrak{k}(E_n))$ and $\mathfrak{k}(\mathrm{Cl}_n)$ are isomorphic Lie algebras. We also recover these classic facts:

$$\mathfrak{k}(E_3) \cong \mathfrak{k}(A_2 \oplus A_1) \cong \mathfrak{u}(2)$$
$$\mathfrak{k}(E_4) \cong \mathfrak{k}(A_4) \cong \mathfrak{so}(5) \cong \mathfrak{sp}(2)$$
$$\mathfrak{k}(E_5) \cong \mathfrak{k}(D_5) \cong \mathfrak{so}(5) \oplus \mathfrak{so}(5) \cong \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$$
$$\mathfrak{k}(E_6) \cong \mathfrak{sp}(4)$$
$$\mathfrak{k}(E_7) \cong \mathfrak{su}(8)$$
$$\mathfrak{k}(E_8) \cong \mathfrak{so}(16)$$



A spin representation of E_n on Cl_n

- $\mathrm{Cl}_n = \left\langle v_1, \ldots, v_n \,\middle|\, v_i^2 = 1,\ v_i v_j = -v_j v_i \right\rangle$
- $A(E_n) = \langle Y_1, \ldots, Y_n \mid Y_i^2 = -1,\ \ldots \rangle$



Lemma 6

There is an algebra isomorphism $\tilde{\rho} : A(E_n) \to \mathrm{Cl}_n$ extending

$$Y_1 \mapsto v_1 v_2, \quad Y_2 \mapsto v_1 v_2 v_3, \quad Y_j \mapsto v_{j-1} v_j \quad \text{for } 3 \le j \le n$$

- Bijective since $\dim \mathrm{Cl}_n = 2^n = \dim A(E_n)$
- $\tilde{\rho} \circ \rho_{max} : \mathfrak{k}(E_n) \to \mathrm{Cl}_n$ is spin representation
- $\mathfrak{m} := \mathrm{im}(\tilde{\rho} \circ \rho_{max})$ compact $\rightsquigarrow$ $\mathfrak{m} \subseteq \mathfrak{k}(\mathrm{Cl}_n)$
- Goal: Show $\dim \mathfrak{m} \ge \dim \mathfrak{k}(\mathrm{Cl}_n)$

A lower bound on the dimension of $\mathfrak{m}$

$$\mathfrak{m} = \langle v_1 v_2 v_3,\ v_1 v_2,\ v_2 v_3,\ \ldots \rangle \le \mathrm{Cl}_n = \left\langle v_1, \ldots, v_n \,\middle|\, v_i^2 = 1,\ v_i v_j = -v_j v_i \right\rangle$$

Lemma 7

Let $n \ge 3$. Then $\mathfrak{m}$ contains all products $v_{j_1} v_{j_2} \cdots v_{j_k}$, where $1 \le j_1 < j_2 < \cdots < j_k \le n$ and $k \equiv 2, 3 \pmod 4$, except possibly for $v_1 v_2 \cdots v_n$ when $n \equiv 3 \pmod 4$.

- Pairs $v_i v_{i+1}$ generate subalgebra $\mathfrak{so}(n) \rightsquigarrow v_i v_j \in \mathfrak{m}$ for $1 \le i < j \le n$
- $\mathfrak{m}$ contains one word of length k, then all: Clear for $k = n$, else pick $j_1, \ldots, j_{k+1}$ pairwise distinct $\Rightarrow [v_{j_1} v_{j_2}, v_{j_2} v_{j_3} \cdots v_{j_{k+1}}] = 2 v_{j_1} v_{j_3} \cdots v_{j_k}$
- $k \equiv 3 \pmod 4$, $k + 3 \le n$: $[v_1 v_2 v_3, \underbrace{v_4 v_5 \cdots v_{k+3}}_{k}] = 2 \underbrace{v_1 v_2 v_3 v_4 \cdots v_{k+3}}_{k+3}$
- $k \equiv 2 \pmod 4$, $k + 2 \le n$: $[v_1 v_2 v_3, \underbrace{v_3 v_4 \cdots v_{k+2}}_{k}] = 2 \underbrace{v_1 v_2 v_4 \cdots v_{k+2}}_{k+1}$

Counting elements

- For $0 \leq k \leq 3$ let $\delta_k : \mathbb{N} \to \mathbb{N} : n \mapsto \binom{n}{k} + \binom{n}{4+k} + \binom{n}{8+k} + \dots$
- $n \geq 3 \rightsquigarrow \dim \mathfrak{m} \geq \begin{cases} \delta_2(k) + \delta_3(k) & \text{if } n \not\equiv 3 \pmod 4 \\ \delta_2(k) + \delta_3(k) - 1 & \text{if } n \equiv 3 \pmod 4 \end{cases}$
- $a, n \in \mathbb{N} \Rightarrow (1 + i^a)^n = \sum_{k=0}^{3} i^{ak} \delta_k(n)$
- $$\begin{aligned} a = 0 &: \delta_0(n) + \delta_1(n) + \delta_2(n) + \delta_3(n) = (1+1)^n = 2^n \\ a = 1 &: \delta_0(n) + i\delta_1(n) - \delta_2(n) - i\delta_3(n) = (1+i)^n = 2^{\frac{n}{2}} \cdot e^{\frac{n2\pi i}{8}} \\ a = 2 &: \delta_0(n) - \delta_1(n) + \delta_2(n) - \delta_3(n) = (1-1)^n = 0 \\ a = 3 &: \delta_0(n) - i\delta_1(n) - \delta_2(n) + i\delta_3(n) = (1-i)^n = 2^{\frac{n}{2}} \cdot e^{-\frac{n2\pi i}{8}} \end{aligned}$$
- Solving system of equations yields explicit lower bounds on $\dim \mathfrak{m}$
- For $n \geq 4$: bound equals $\dim \mathfrak{k}(\mathrm{Cl}_n) \rightsquigarrow \dim \mathfrak{k}(\mathrm{Cl}_n) = \dim \rho_{max}(\mathfrak{k}(E_n))$

Cartan-Bott periodicity for $\rho_{max}(\mathfrak{k}(E_n))$ revisited

Theorem 8 (H.-Köhl 2014)

For $n \geq 4$, the isomorphism types of $\rho_{max}(\mathfrak{k}(E_n))$ are as follows:

n mod 8	$\rho_{max}(\mathfrak{k}(E_n))$
0	$\mathfrak{so}(2^{\frac{n}{2}})$
1	$\mathfrak{so}(2^{\frac{n-1}{2}}) \oplus \mathfrak{so}(2^{\frac{n-1}{2}})$
2	$\mathfrak{so}(2^{\frac{n}{2}})$
3	$\mathfrak{su}(2^{\frac{n-1}{2}})$
4	$\mathfrak{sp}(2^{\frac{n-2}{2}})$
5	$\mathfrak{sp}(2^{\frac{n-3}{2}}) \oplus \mathfrak{sp}(2^{\frac{n-3}{2}})$
6	$\mathfrak{sp}(2^{\frac{n-2}{2}})$
7	$\mathfrak{su}(2^{\frac{n-1}{2}})$

Outlook: Integrating spin representations to groups

- Spin representation of $\mathfrak{so}(n)$ integrates to representation of $\mathrm{Spin}(n)$

- $$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathrm{Spin}(n) & \longrightarrow & \mathrm{SO}(n) & \longrightarrow & 1 \\ & & & & \rho \big\downarrow & & & & \\ & & & & \mathfrak{so}(n) & & & & \end{array}$$

- "Integrate" spin representations to 2^k-fold covers of "maximal compact" subgroup $K(\Pi)$ of Kac-Moody group $G(\Pi)$, where $k \geq 0$ depends on Π

- $$\begin{array}{ccccccccc} 1 & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^k & \longrightarrow & \mathrm{Spin}(\Pi) & \longrightarrow & K(\Pi) & \longrightarrow & 1 \\ & & & & \rho \big\downarrow & & & & \\ & & & & \mathfrak{k}(\Pi) & & & & \end{array}$$

- $\rightsquigarrow K(\Pi)$ in general is not simple

The End

Thank you for your attention!