Endotrivial modules for some "Very Important Groups":

An endotrivial patchwork

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Outline

About today's highlights

The following results have been obtained in collaboration with

- Jon Carlson and Daniel Nakano
- Caroline Lassueur

Outline

Setting

- p a prime
- $k = \bar{k}$ a field of characteristic p
- G a finite group with $|G| \equiv 0 \pmod p$
- kG-module = finitely generated left kG-module
- $M^* = \mathrm{Hom}_k(M, k)$

Definition

M is an **endotrivial kG-module** if $\mathrm{End}_k M \cong k \oplus$ (proj) .
The **group of endotrivial modules** is the group of stable isomorphism classes $T(G)$ of endotrivial kG-modules, with

$$[M] + [N] = [M \otimes_k N] \quad \text{for all} \quad [M],\ [N] \in T(G)$$

Example

$0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$ exact with L projective. Then

$$M \text{ is endotrivial } \iff N \text{ is endotrivial}$$

Proof in exercise. Hint: use $\otimes$, $(\ )^*$, *Schanuel's lemma and Krull-Schmidt theorem.*

In particular,

$$\ldots,\ \Omega^{-1}(k)\ ,\ k\ ,\ \Omega(k)\ ,\ \Omega^2(k)\ ,\ldots$$

are endotrivial, and

$$M^* = \mathrm{Hom}_k(M,k) \text{ is endotrivial } \iff M \text{ is endotrivial}$$

In fact,

$$[M^*] = -[M] \text{ in } T(G)$$

Ubiquitous results

- $T(G)$ is finitely generated: $\boxed{T(G) = TT(G) \oplus TF(G)}$
- $TF(G) \cong \mathbb{Z}^n$ where n is equal to
    - $\sharp$ conj. cl. max sgps $C_p \times C_p$ of G if G has p-rank ≤ 2
    - that number plus one if G has p-rank at least 3

  If $n \geq 1$, then $[\Omega(k)]$ spans an infinite cyclic direct summand of $T(G)$.
- Let $N = N_G(P)$ where $P \in \mathrm{Syl}_p(G)$. Green correspondence implies:

  $$\mathrm{Res}^G_N \ : \ T(G) \to T(N)$$

  is an injective group homomorphism. Moreover,

  $$T(N) = \langle \text{ generators } \mid \text{ relations } \rangle \quad \text{is “known” and}$$

  $TT(N) = X(N) \cong N/N'P$ is the group of 1-dimensional kN-modules unless P cyclic, generalised quaternion or semi-dihedral.

Obstacles to the classification

The quest for the classification of endotrivial modules has been ongoing for the past 40 years ca. There are two main issues that still cannot be answered:

Obstacle 1

Generators for $TF(G)$ when $TF(G) \cong \mathbb{Z}^n$ and $n > 1$?
All that we know is that $n \leq p+1$ if $p > 2$ and $n \leq 5$ if $p = 2$, and that both bounds are optimal.

Obstacle 2

Kernel of $\mathrm{Res}^G_P : T(G) \to T(P)$. That is, what are the trivial source endotrivial modules?
"Easy extreme cases": when P is TI or $O_p(G) > 1$.

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Outline

1. Endotrivial modules
   - A brief introduction
   - Today's focus
   - More on "the" torsion-free part
   - More on the torsion subgroup
2. A blend of new methods to study $TT(G)$
   - Using character theory
   - Using the local structure
   - Using the determinant map

Today's Very Important Groups

Today's talk presents some applications of the "latest techniques" together with traditional methods to classify endotrivial modules for some "important" groups.

(i) Finite groups of Lie type A in nondefining characteristic, i.e. subquotients of some $\mathrm{GL}(n, q)$ with $(p, q) = 1$ (*with Carlson and Nakano*)

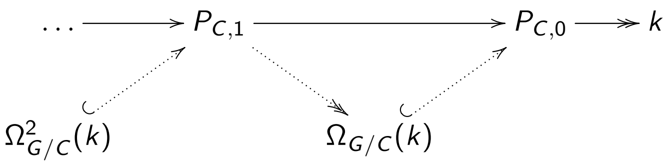
(ii) Sporadic simple groups and their Schur covers (*with Lassueur*)

(iii) Schur covers of alternating and symmetric groups (*with Lassueur*)

Outline

$TF(G)$: extraspecial considerations - Part I

Suppose p odd and $G = p_+^{1+2}$. For each noncentral subgroup C of G of order p, let



be a minimal relative C-projective resolution of k. Then

$$\Omega^2_{G/C}(k) \text{ is endotrivial and}$$
$$TF(G) = \langle [\Omega_G(k)], [\Omega^2_{G/C_1}(k)], \dots, [\Omega^2_{G/C_{p+1}}(k)] \rangle \cong \mathbb{Z}^{p+1}$$

where the elements are subject to the relation

$$2[\Omega_G(k)] = \sum_{1 \le i \le p+1} [\Omega^2_{G/C_1}(k)] \text{ in } T(G).$$

$TF(G)$: extraspecial considerations - Part II

Suppose $G = F_1$ is the *Friendly Giant* (i.e. Fischer-Griess “Monster”). Then

$$|G| = 2^{46}3^{20}5^9 7^6 11^2 13^3 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

and $P = 13_+^{1+2} \in \mathrm{Syl}_{13}(G)$. By Ruiz-Viruel's classification of p-local finite groups over p_+^{1+2} we know that

$$N_G(P)/PC_G(P) \cong P \rtimes (C_4.\mathfrak{S}_4)$$

and the 14 elementary abelian subgroups of P of order 9 fusent in G into 2 conjugacy classes (one class of size 6 formed by all the 13-centric 13-radical elementary abelian subgroups of P of order 9, and the remaining 8 other form the other class). So

$$TF(G) = \langle [\Omega_G(k)]\ ,\ [?] \rangle \cong \mathbb{Z}^2$$

Challenge: determine [?].

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$TT(G)$: what we know

Let $P \in \mathrm{Syl}_p(G)$ and $N = N_G(P)$. Write $X(N)$ for the multiplicative group (for $\otimes$) of iso classes of 1-dim kN-modules. Recall $X(N) \cong N/N'P$ where $N' = [N, N]$ is the derived subgroup of N.

Strategy: use $TT(N)$ to find $TT(G)$ via Green correspondence.

(i)
$$TT(P) \cong \begin{cases} \mathbb{Z}/2 & P \text{ cyclic of order} > 2\text{, or } P \text{ semi-dihedral} \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 & P \text{ generalised quaternion} \\ 0 & \text{otherwise} \end{cases}$$

(ii) $0 \longrightarrow X(N) \longrightarrow TT(N) \longrightarrow TT(P) \longrightarrow 0$ is exact.

(iii) M indecomposable endotrivial kN-module $\Longleftrightarrow \mathrm{Res}^N_P M$ indecomposable endotrivial kP-module.

(iv) $\mathrm{Res}^G_P : TT(G) \hookrightarrow TT(N)$ is an injective group homomorphism.

So what is “the ultimate” question ?

THE QUESTION IS:

What are the trivial source endotrivial kG-modules?
That is, find

$$K(G) := \mathrm{Ker}\left(\mathrm{Res}^G_P : TT(G) \to TT(P)\right)$$

The easy cases

1. If $O_p(G) > 1$, then

   $$K(G) = X(G) \quad \text{the group of iso classes of 1-dimensional } kG\text{-modules.}$$

2. If P is TI (i.e. $P^g \cap P = 1$ for all $g \in G - N$), then

   $$K(G) \overset{\mathrm{Res}^G_N}{\cong} K(N) = X(N) \cong N/PN'$$

Outline

$TT(G)$: a method with "Character"

Theorem (Lassueur, Malle, Schulte (2013) and Lassueur, Malle (2015))

Let $(k, \mathcal{O}, K)$ be a p-modular system large enough. Let $P \in \mathrm{Syl}_p(G)$ and $N = N_G(P)$.

(i) *Let V be an endotrivial kG-module. Then V lifts to a $\mathcal{O}G$-lattice $\widehat{V}$.*

(ii) *Let M be the kG-Green correspondent of a 1-dim kN-module. Put $\widehat{M}$ for the (unique) indecomposable trivial source $\mathcal{O}G$-lattice lifting M and χ for the ordinary character afforded by $\widehat{M}$. Then M is endotrivial if and only if $\chi(u) = 1$ for all $1 \neq u \in P$ and $|\chi(g)| = 1$ for each p-singular element $g \in G$.*

Application

All finite groups whose character table is known.

An idiosyncratic instance

Let $p = 3$ and $G = J_2$, so that $2.G = 2.J_2$. Then $|G| = 2^7 3^3 5^2 7$ with $P = 3_+^{1+2}$ and $N = P \rtimes C_8$.
We have $T(P) \cong \mathbb{Z}^4$ by the "extraspecial example" above.
These 4 conj classes fuse in G, so that

$$TF(G) = \langle [\Omega_G(k)] \rangle \cong \mathbb{Z}$$

Moreover,

$$TT(G) = \{[M] \mid M \quad \text{trivial source endotrivial } kG\text{-module } \} \text{ and}$$

$$TT(G) \xrightarrow{\mathrm{Res}_N^G} TT(N) \cong \mathbb{Z}/8 \quad \text{is injective.}$$

An idiosyncratic instance, cont'd I

Let χ be a 1-dimensional kN-module. We need to determine if the kG-Green correspondent M_χ of χ is endotrivial.
Using the GAP character tables, we induce the linear characters of N.
Because the principal block of kG is the unique block with defect group P, we can "cut" by the corresponding idempotent e_0. That is, ask GAP to calculate

$$e_0 \cdot \mathrm{Ind}_N^G \chi \quad \text{for } \chi \in X(N) = \{(1_1), 1_2, \dots, 1_8\}$$

and then look for the Green correspondent of χ.
By Theorem 2.1 and a "well known" result (Landrock-Scott), it is enough to look at the character values (and degrees). We find that only the Green correspondent $\Gamma_G(1_7)$ of the nontrivial selfdual character $1_7 \in X(N)$ affords the ordinary character $\chi_4 + \chi_5 + \chi_{13}$ of an endotrivial kG-module. So,

$$TT(G) = \langle [\Gamma_G(1_7)] \rangle \cong \mathbb{Z}/2$$

An idiosyncratic instance, cont'd II

For $2.G = 2.J_2$ and $p = 3$, we know that

$$\mathrm{Inf}_G^{2.G} : T(G) \hookrightarrow T(2.G)$$

So, left to answer: “Does $2.G$ have faithful endotrivial modules?” Inspection of the character tables and Theorem 2.1 show that there is no faithful character of $2.G$ that takes value ± 1 on all the 3-singular elements. Therefore,

$$\mathrm{Inf}_G^{2.G} : T(G) \longrightarrow T(2.G) \text{ is an isomorphism.}$$

Outline

$TT(G)$: a local "heuristic"

Original credits: Balmer, Carlson and Thévenaz.

Definition

$\forall\ 1 < Q < P \in \mathrm{Syl}_p(G)$ define

1. $\rho^1(Q) = N_G(Q)'$ and $\rho^i(Q) = \langle N_G(Q) \cap \rho^{i-1}(R) \mid 1 < R \leq P \rangle$ for $i > 1$.
2. $K(G) =$ subgroup of $T(G)$ formed by the iso classes of trivial source endotrivial kG-modules.

In particular, the group of 1-dim kG-modules $X(G)$ is a subgroup of $K(G)$.

Theorem (Carlson, Thévenaz (2015))

If there exist some subgroup $1 < Q \trianglelefteq P \in \mathrm{Syl}_p(G)$ and an integer $i \geq 1$ such that $N_G(P) \leq N_G(Q)$ and $\rho^i(Q) = N_G(Q)$, then $K(G) = \{[k]\}$.

Local adaptation

Adaptation of the previous theorem to the specific context: finite groups of Lie type A.

Theorem

Let $P \in \mathrm{Syl}_p(G)$. Suppose there exists H with $N_G(P) \leq H \leq G$ which satisfies:

1. $K(H) = X(H)$
2. $H = \langle g_1, \ldots, g_m \rangle$ *such that for each i, either*
   - (i) $g_i \in H'P$, *or*
   - (ii) $\exists\ H_i \leq G$ *s.t. the following holds:*
     - (a) $K(H_i) = X(H_i)$,
     - (b) $|H_i \cap H| \equiv 0 \pmod{p}$, *and*
     - (c) $g_i \in H_i'$

Then $K(G) = \{[k]\}$.

A special instance

Let $G = \mathrm{SL}(re, q)$ and $P \in \mathrm{Syl}_p(G)$, where

- $q^e - 1 \equiv 0 \pmod p$ with $1 < e < p$, and
- $r = ap^s$ with $2 < a < p$

We want to show that $K(G) = \{[k]\}$, i.e. $TT(G) = \{[k]\}$. Take the Levi subgroup $\widehat{L} \cong \mathrm{GL}(p^s e, q)^a < \mathrm{GL}(re, q)$ and $L = \widehat{L} \cap G$.
Then $N_G(P) \leq N_G(L)$,

$$1 \longrightarrow L \longrightarrow N_G(L) \longrightarrow \mathfrak{S}_a \longrightarrow 1$$

and

$$N_G(L) = \langle g_1, g_2, \ldots, g_m \rangle \quad \text{with} \quad N_G(L)' = \langle g_2, \ldots, g_m \rangle$$

So $N_G(L)/N_G(L)' = \langle g_1 N_G(L)' \rangle \cong C_2$ and $g_1 = \begin{pmatrix} & I_{p^s e} & \\ -I_{p^s e} & & \\ & & I_{(a-2)p^s e} \end{pmatrix}$.

A special instance, cont'd

To show that $K(G) = \{[k]\}$, we want to show that $N_G(L)$ satisfies the conditions of H as in the theorem. Since $N_G(L)/N_G(L)' \cong \langle g_1 N_G(L)' \rangle$, it is enough to show that g_1 is in some subgroup H_1' of order divisible by p and that $K(H_1) = X(H_1)$.Take the Levi subgroup $\widehat{H}_1 \cong \mathrm{GL}(2p^s e, q) \times \mathrm{GL}((a-2)p^s e, q) < \mathrm{GL}(re, q)$ and put $H_1 = \widehat{H}_1 \cap G$.We know that $K(H_1) = X(H_1)$ (exercise using direct product decomposition of H_1 and the fact that the two factors have orders divisible by p.)
Now $H_1' \cong \mathrm{SL}(2p^s e, q) \times \mathrm{SL}((a-2)p^s e, q)$ and so $g_1 \in H_1'$, as required.
Similar arguments lead to:

Theorem

Let $q^e - 1 \equiv 0 \pmod p$ with $1 \leq e < p$, and suppose that $n \geq 2e > 2$ and that $p > 2$. Then, for any group G with $\mathrm{SL}(n,q) \leq G \leq \mathrm{GL}(n,q)$,

$$T(G) = \{[\Omega_G(k)]\} \oplus X(G) \cong \mathbb{Z} \oplus X(G)$$

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Tackling quotients: the exceptions

Goal: extend the results on $T(G)$ to $T(G/Z)$ for $Z \leq Z(G)$ for $\mathrm{SL}(n,q) \leq G \leq \mathrm{GL}(n,q)$ where $q^e - 1 = p^t d$ with $(p,d) = 1 \leq t$ and $1 \leq e < p$.

The previous method can be used to show

$$T(G/Z) = \{[\Omega_{G/Z}(k)]\} \oplus X(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$$

except when $O_p(G/Z) = 1$ and either

(i) $n = p = 3$ and $e = 1$, e.g. $G = \mathrm{SL}(3,7)$, or

(ii) $p = 2$ and $n = 2,3$ in "most cases", e.g. $\mathrm{SL}(3,7)$.

In (i), $P \in \mathrm{Syl}_3(G)$ is a normal subgroup of $C_{3^t} \wr C_3$ of index at most 3^t, and $1 \leq Z \leq C_{3^t}$.

In (ii), the exceptions are when $P \in \mathrm{Syl}_2(G)$ is semi-dihedral or wreathed. If $G = \mathrm{GL}(3,q)$, then P has rank 3, a nontrivial normal 2-subgroup and no maximal Klein subgroup, so that $T(G) \cong \mathbb{Z} \oplus X(G)$.

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Tackling quotients: the exceptions

Goal: extend the results on $T(G)$ to $T(G/Z)$ for $Z \leq Z(G)$ for $\mathrm{SL}(n,q) \leq G \leq \mathrm{GL}(n,q)$ where $q^e - 1 = p^t d$ with $(p,d) = 1 \leq t$ and $1 \leq e < p$.

The previous method can be used to show

$$T(G/Z) = \{[\Omega_{G/Z}(k)]\} \oplus X(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$$

except when $O_p(G/Z) = 1$ and either

(i) $n = p = 3$ and $e = 1$, e.g. $G = \mathrm{SL}(3,7)$, or

(ii) $p = 2$ and $n = 2, 3$ in "most cases", e.g. $\mathrm{SL}(3,7)$.

In (i), $P \in \mathrm{Syl}_3(G)$ is a normal subgroup of $C_{3^t} \wr C_3$ of index at most 3^t, and $1 \leq Z \leq C_{3^t}$.

In (ii), the exceptions are when $P \in \mathrm{Syl}_2(G)$ is semi-dihedral or wreathed.

If $G = \mathrm{GL}(3,q)$, then P has rank 3, a nontrivial normal 2-subgroup and no maximal Klein subgroup, so that $T(G) \cong \mathbb{Z} \oplus X(G)$.

Tackling quotients: determinant and factorisations

Preliminary remark

If $e > 1$, then $|Z| \not\equiv 0 \pmod{p}$, and $\mathrm{Inf}_{G/Z}^{G} : T(G/Z) \to T(G) \cong \mathbb{Z} \oplus X(G)$ is injective. Thus $T(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$.

Let $\det : G \to \mathbb{F}_q^\times$, where $SL(n,q) \le G \le \mathrm{GL}(n,q)$ for $n = 2, 3$. Note that $Z(G) = \{aI_n \mid a^n \in \det(G)\}$.

Lemma

(i) *If $p = n = 3$, then $G/Z \cong H \times V$ where $H = \det^{-1}(Q)$ for $Q \in \mathrm{Syl}_3(\det(G/Z))$ and $|V| \not\equiv 0 \pmod{3}$.*

(ii) *If $p = n = 2$, then $G/Z \cong H \times V$ where $H = \det^{-1}(Q)$ for $Q \in \mathrm{Syl}_2(\det(G/Z))$ and $|V|$ is odd.*

(iii) *If $p = 2 < 3 = n$, then $G/Z \cong H \times V \times W$ where $H = \det^{-1}(Q)$ for $Q \in \mathrm{Syl}_3(\det(G/Z))$, where V is a cyclic 2-group and $(|W|, 6) = 1$.*

Application

RECALL: we want to find $T(G/Z)$. Here the focus is on the group structure of G/Z.
Lemma 2.6 allows us to reduce the analysis of the numerous possibilities for quotients of intermediary subgroups $\mathrm{SL}(n,q) \leq G \leq \mathrm{GL}(n,q)$ to one of the groups $\mathrm{PGL}(n,q)$ or $\mathrm{PSL}(n,q)$ in the outcast cases for $n,p = 2,3$.

Proposition

Suppose that $p = n = 3$.

(i) *If $\frac{q-1}{|\det(G)|} \not\equiv 0 \pmod 3$, then $G/Z \cong \mathrm{PGL}(3,q) \times V$ where $V \cong \det(G)/\det(Z)$.*

(ii) *Otherwise $G/Z \cong \mathrm{PSL}(3,q) \times V$, where V is the 3-complement in $\det(G)/\det(Z)$.*

Mutatis mutandis for $p = 2$ (Exercise for the audience. Hint: there are a few more cases). The consequence for $p = 3$ is as follows:

Application, cont'd

Theorem (CMN (2015))

Let $p = 3$ *and* $\mathrm{SL}(n, q) \leq G \leq \mathrm{GL}(n, q)$ *as above, and let* V *be the 3-complement in the cyclic group* $\det(G)/\det(Z)$.

(i) *If* $\frac{q-1}{|\det(G)|} \not\equiv 0 \pmod 3$, *then* $T(G/Z) \cong \mathbb{Z}^3 \oplus V$ *where* $V \cong \det(G)/\det(Z)$.

(ii) *Otherwise,* $\frac{q-1}{|\det(G)|} \equiv 0 \pmod 3$ *and:*

1. *If* $q \equiv 1 \pmod 9$, *then* $T(G/Z) \cong \mathbb{Z}^4 \oplus V$.
2. *If* $q \equiv 4, 7 \pmod 9$, *then* $T(G) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus V$.

"Fatty" quotients

RECALL: we want to find $T(G/Z)$. Here the focus is on $TF(G/Z)$.

Lemma

Let $\mathrm{SL}(2,q) \leq G \leq \mathrm{GL}(2,q)$ *with* $q-1 \equiv 0 \pmod 4$ *and* $1 < Z < Z(G)$ *with* $|Z(G) : Z| \equiv 0 \pmod 2$. *Then a Sylow 2-subgroup* P *of* G/Z *has rank* 3 *and* $Z(P)$ *has rank* 2.

For instance,
$G = \mathrm{GL}(2,5)$ and $p = 2$, so that $P \cong C_4 \wr C_2$ has rank 2. Take $1 < Z < Z(P) \cong C_4$. Then P/Z has rank 3 and $Z(P/Z) \cong C_2 \times C_2$.

Corollary (CMN (2015))

Let $\mathrm{SL}(2,q) \leq G \leq \mathrm{GL}(2,q)$ *with* $q-1 \equiv 0 \pmod 4$ *and* $1 < Z < Z(G)$ *with* $|Z| \equiv 0 \pmod 2$ *and* $|Z(G) : Z| \equiv 0 \pmod 2$. *Then* $TF(G/Z) \cong \mathbb{Z}$.

"Fatty" quotients

RECALL: we want to find $T(G/Z)$. Here the focus is on $TF(G/Z)$.

Lemma

Let $\mathrm{SL}(2,q) \leq G \leq \mathrm{GL}(2,q)$ *with* $q-1 \equiv 0 \pmod 4$ *and* $1 < Z < Z(G)$ *with* $|Z(G):Z| \equiv 0 \pmod 2$. *Then a Sylow 2-subgroup* P *of* G/Z *has rank* 3 *and* $Z(P)$ *has rank* 2.

For instance,
$G = \mathrm{GL}(2,5)$ and $p=2$, so that $P \cong C_4 \wr C_2$ has rank 2. Take $1 < Z < Z(P) \cong C_4$. Then P/Z has rank 3 and $Z(P/Z) \cong C_2 \times C_2$.

Corollary (CMN (2015))

Let $\mathrm{SL}(2,q) \leq G \leq \mathrm{GL}(2,q)$ *with* $q-1 \equiv 0 \pmod 4$ *and* $1 < Z < Z(G)$ *with* $|Z| \equiv 0 \pmod 2$ *and* $|Z(G):Z| \equiv 0 \pmod 2$. *Then* $TF(G/Z) \cong \mathbb{Z}$.

"Fatty" quotients

RECALL: we want to find $T(G/Z)$. Here the focus is on $TF(G/Z)$.

Lemma

Let $\mathrm{SL}(2,q) \leq G \leq \mathrm{GL}(2,q)$ *with* $q-1 \equiv 0 \pmod 4$ *and* $1 < Z < Z(G)$ *with* $|Z(G):Z| \equiv 0 \pmod 2$. *Then a Sylow 2-subgroup P of G/Z has rank 3 and $Z(P)$ has rank 2.*

For instance,
$G = \mathrm{GL}(2,5)$ and $p = 2$, so that $P \cong C_4 \wr C_2$ has rank 2. Take $1 < Z < Z(P) \cong C_4$. Then P/Z has rank 3 and $Z(P/Z) \cong C_2 \times C_2$.

Corollary (CMN (2015))

Let $\mathrm{SL}(2,q) \leq G \leq \mathrm{GL}(2,q)$ *with* $q-1 \equiv 0 \pmod 4$ *and* $1 < Z < Z(G)$ *with* $|Z| \equiv 0 \pmod 2$ *and* $|Z(G):Z| \equiv 0 \pmod 2$. *Then* $TF(G/Z) \cong \mathbb{Z}$.

The end. . .

Thanks!