Effective algorithms for groups of Lie type

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Overview of lecture

G “large" finite group described by generating set X.
e.g. $G = \langle X \rangle \leq \mathrm{GL}(\mathrm{d}, \mathrm{q})$ or $G = \langle X \rangle \leq \mathrm{Sym}(n)$.

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- Explain the model.
- Discuss how to construct the model.

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Cannon, Holt et al. (1997–): use $\mathcal{C}$ in practical algorithms.

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Also compute series

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_r = L \lhd G$$

where $N_i \trianglelefteq G$ and N_i/N_{i-1} is elementary abelian.

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In many cases, easy to reduce the computation for TF-group H to almost simple groups.

Examples of algorithms using Soluble Radical model

- Determine conjugacy classes of elements of G; (Cannon & Souvignier, 1997)
- Determine maximal subgroups of G; (Cannon & Holt, 2004) and (Eick & Hulpke, 2001)
- Determine the automorphism group of G; (Cannon & Holt, 2003)
- Determine conjugacy classes of subgroups of G; (Cannon, Cox & Holt, 2001)

How do we construct the characteristic chain?

Basic approach: Schreier-Sims techniques, developed first in permutation group context.

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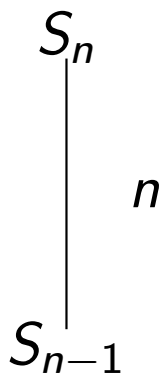

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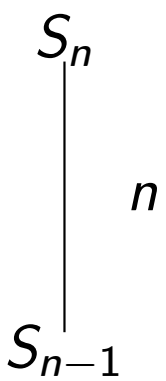

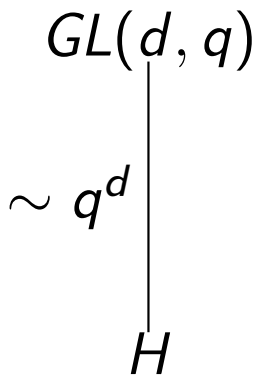

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- G preserves some natural linear structure associated with the action of G on V, and has normal subgroup related to this structure,

  OR

- G is almost simple modulo scalars: $T \leq G/Z \leq Aut(T)$ where T is simple. e.g. $G = \mathrm{SL}(d, q)$, invertible matrices of determinant 1.

Geometry following Aschbacher: general strategy

1. Determine (at least one of) its Aschbacher categories.
2. If $N \lhd G$ exists, recognise N and G/N recursively, ultimately obtaining a composition series for the group.

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7 categories giving normal subgroup

Prototype: G acts imprimitively on V

G preserves decomposition of V as direct sum

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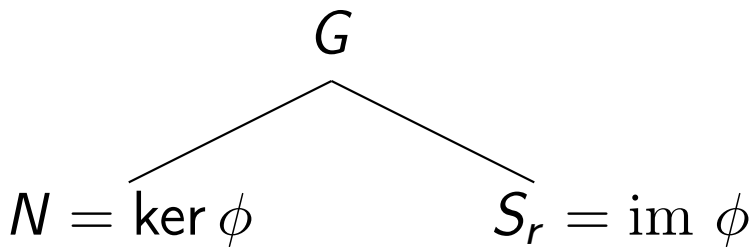
Then $\phi : G \to S_r$ where $r \leq d$ and $N = \ker \phi$.



Holt, Leedham-Green, O'B & Rees (1996)

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   - Set up "constructive isomorphisms" between L and its *standard copy*.

Decide membership of category

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After 20 years: membership of 5 of the geometric categories are decidable in polynomial time; other are decidable.

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Liebeck (1985): almost all maximal non-classical subgroups of $\mathrm{GL}(d, q)$ have order at most q^{3d}: much smaller than $O(q^{d^2})$.

Constructive recognition

$C = \langle X \rangle \leq \mathrm{GL}(d, q)$ where C is (quasi)simple.

C is standard or "gold" copy.

C classical: natural copy fixing specific form.

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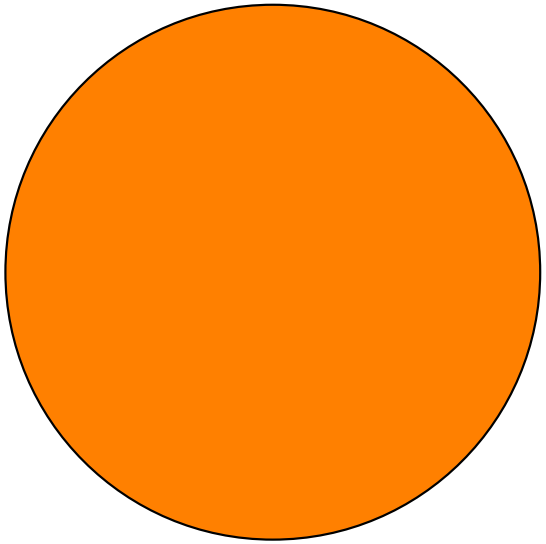
Key idea: use standard generators.

Using standard generators



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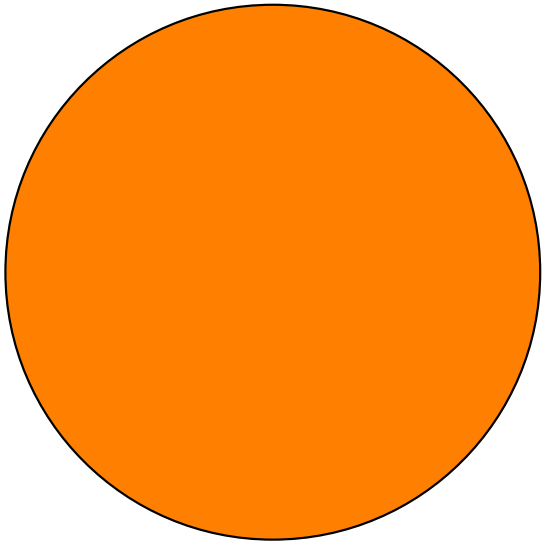

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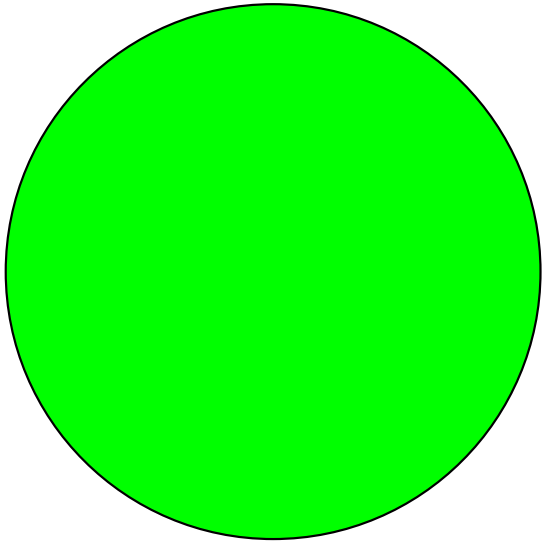

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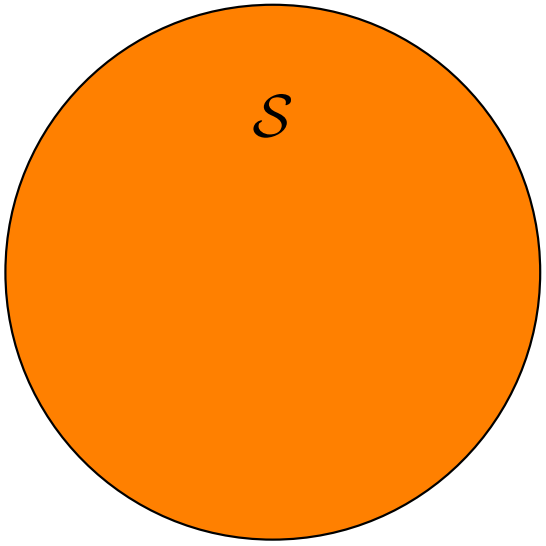

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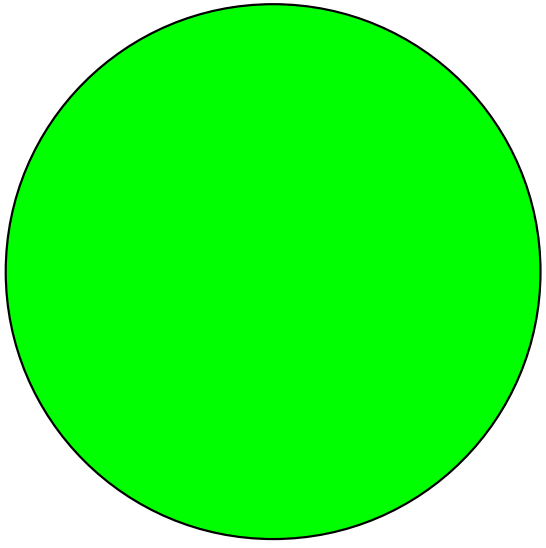

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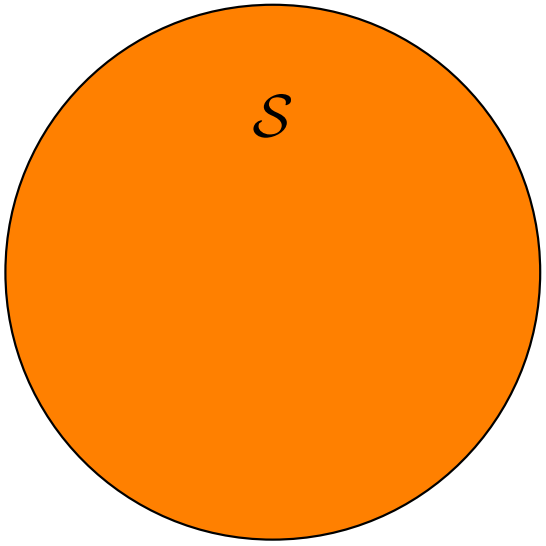

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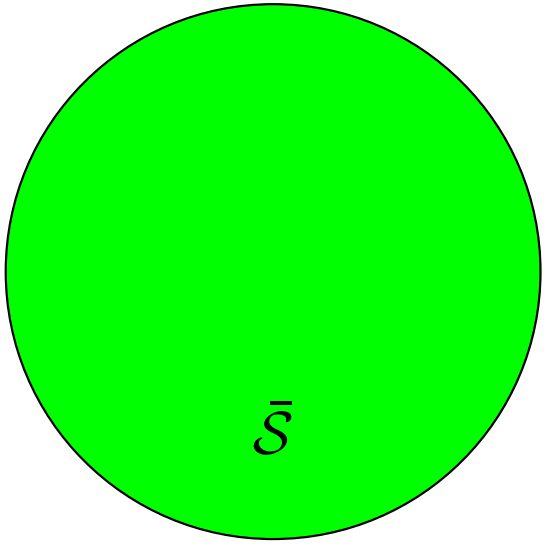

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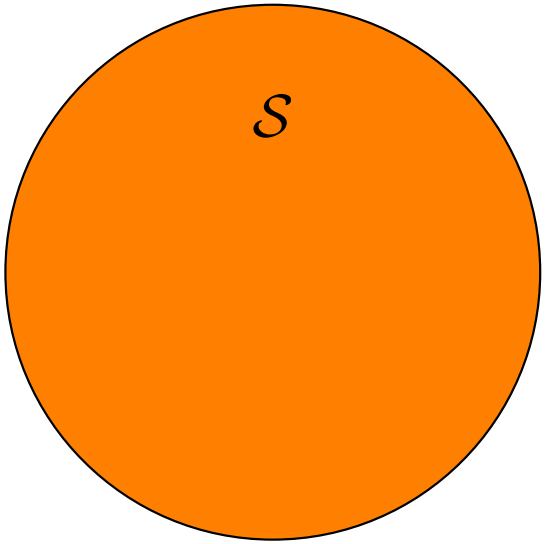

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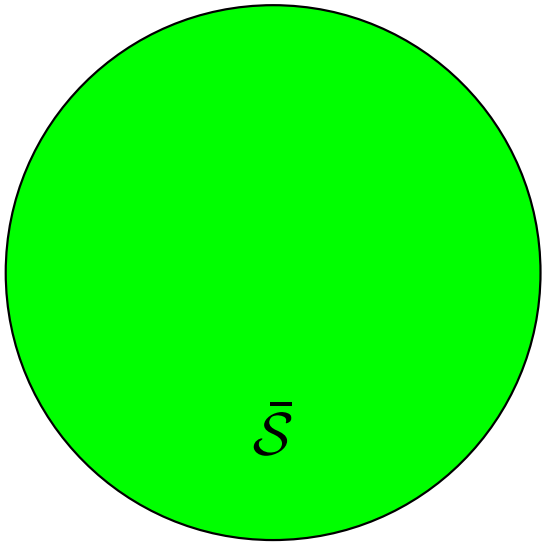

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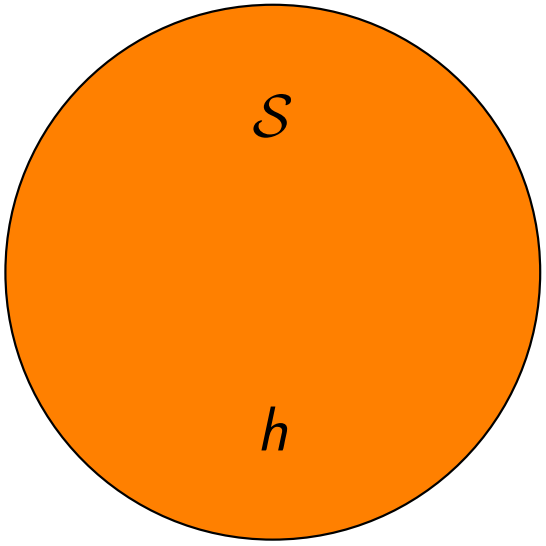

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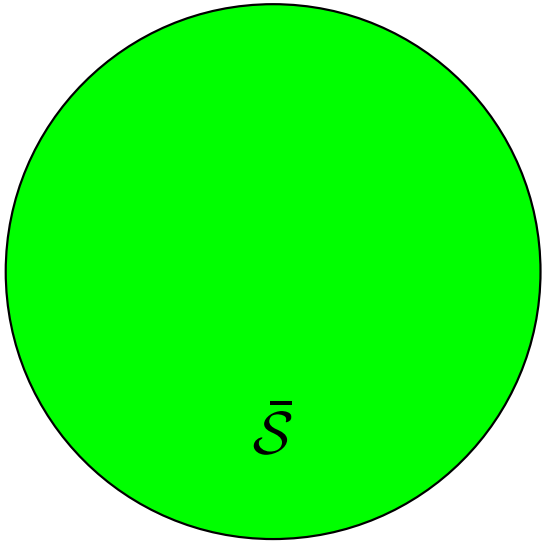

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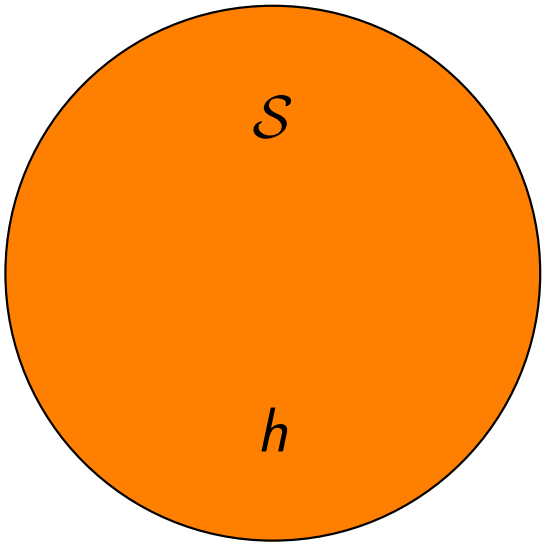

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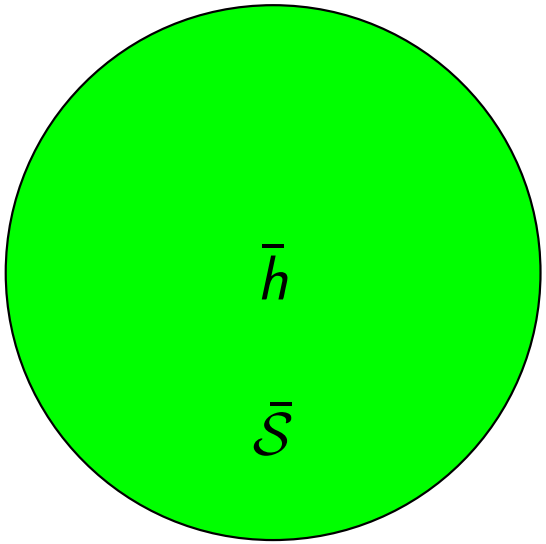

Application I: Maximal subgroups of classical groups

Kleidmann & Liebeck (1990): describe some maximal subgroups of classical groups where $d \geq 13$.

Bray, Holt & Roney-Dougal (2013): construct generating sets for geometric maximal subgroups, and all maximals for $d \leq 12$.

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Use $\phi : C \longmapsto G$ to construct image of M in arbitrary representation G.

Application II: Conjugacy classes of elements

Wall (1963): description of conjugacy classes and centralisers of elements of classical groups.

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Liebeck and O'Brien, under development: algorithm which writes down classes in standard copy of group of Lie type.

Standard generators for $\mathrm{SL}(d, q)$

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s, δ, u lie in copy of $\mathrm{SL}(2, q)$ and act on $\langle e_1, e_2 \rangle$ as:

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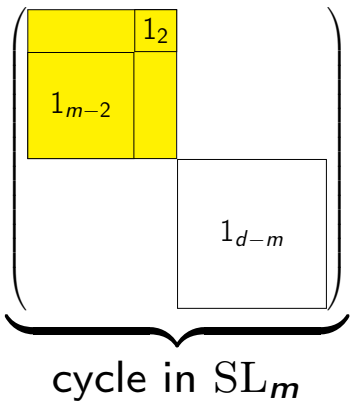

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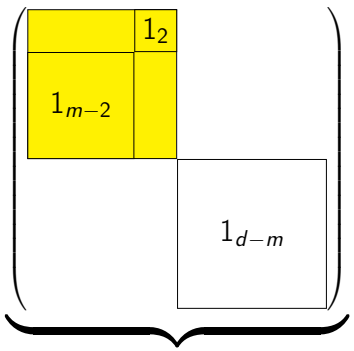

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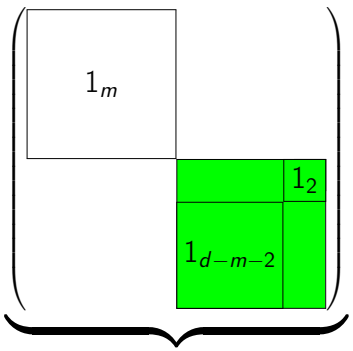

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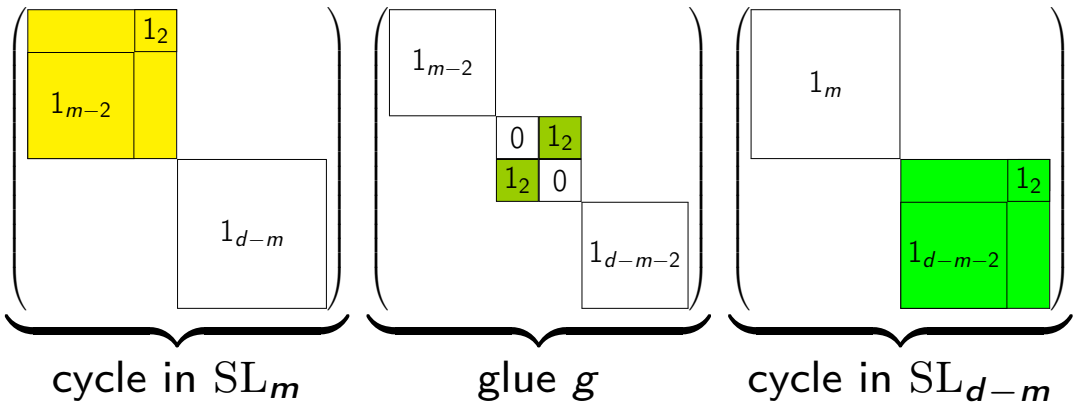

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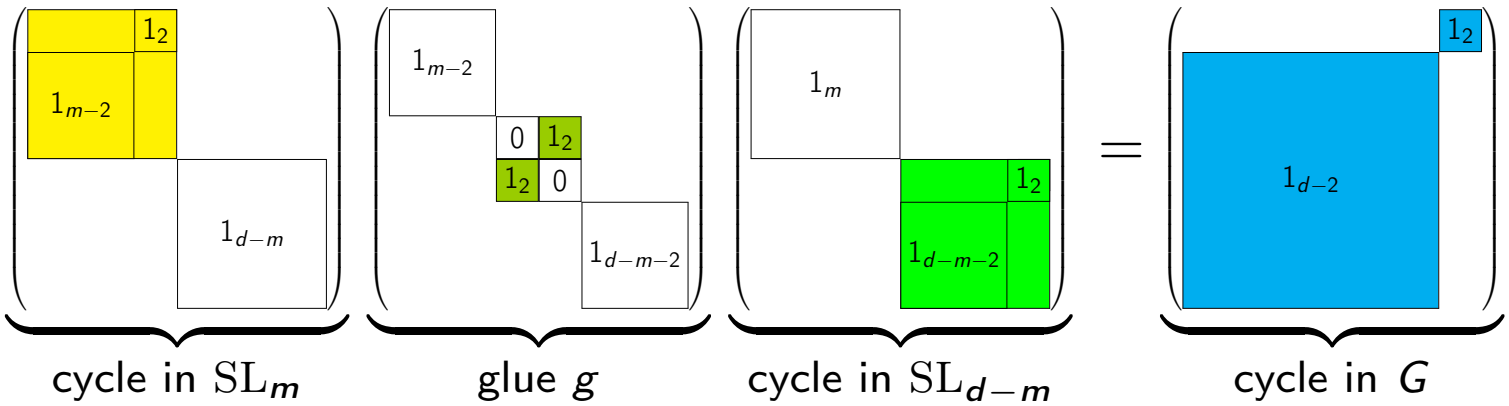

Leedham-Green and O'B, 2009; Dietrich, L-G, Lübeck, O'B, 2013; D, L-G, O'B, 2015

Theorem

There is a Las Vegas algorithm that takes as input $G \cong \mathrm{SX}(d, q) = \langle X \rangle$, and returns standard generators $\mathcal{S}$ for G as words in X. The algorithm has complexity $O(d^4 \log q)$ measured in field operations.

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Bäärnhielm (2006, 2014): Algorithms to construct standard generators for matrix representations of Suzuki, large and small Ree groups.

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- Praeger et al., 2015; In polynomial time, can construct $H \leq G \cong SX(d, q)$ where $H \cong SX(m, q)$ and $m \approx d/2$.

Constructive recognition for other families

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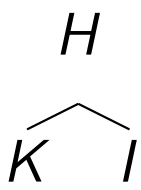

- Node: section H of G.

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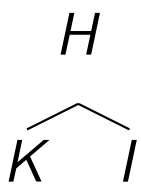

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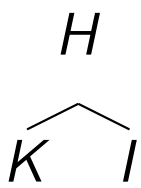

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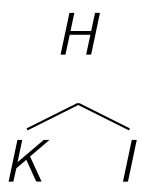

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- Leaf is "composition factor" of G: simple modulo scalars. Cyclic not necessarily of prime order.

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So inductively need to know presentations **only for the leaves** – or composition factors.

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Every non-abelian finite simple group of rank n over $\mathrm{GF}(q)$, with possible exception of Ree groups ${}^2G_2(q)$, has a presentation with a bounded number of generators and relations and total length $O(\log n + \log q)$.

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Output of CompositionTree

Given $G = \langle X \rangle \leq \mathrm{GL}(\mathrm{d}, \mathrm{q})$ as input.
Output:

- a composition series: $1 = G_0 \lhd G_1 \lhd G_2 \cdots \lhd G_m = G$.
- A representation $S_k = \langle X_k \rangle$ of G_k / G_{k-1}
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Construct presentation for group defined by tree and verify that G satisfies the relations.

From composition series to $\mathcal{C}$

Bäärnhielm, Holt, Leedham-Green & O'B (2014): refine composition series obtained from "geometric model" to obtain chief series reflecting characteristic structure.

Holt: developed Soluble Radical model algorithms using tree as infrastructure.

Publicly available in Magma; parts available in GAP.