On Exceptional Permutation Groups of Order p^5

(joint with John R. Britnell and Tony Skyner)

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Minimal Permutation Degree of a Finite Group

The minimal faithful permutation degree, $\mu(G)$ of a finite group G is:

$$\mu(G) := \min\{n \mid \exists\, \sigma : G \hookrightarrow \mathrm{Sym}(n)\}.$$

Easy to define, but very difficult to calculate.
Clearly $\mu(G) \leq |G|$ but this is very inefficient.

Theorem (Johnson, 1971)

$$\mu(G) = |G| \iff G \in \{C_{p^\alpha}, Q_{2^\alpha}, C_2 \times C_2\}.$$

Construction Permutation Representations

Can represent G on a set of right (left) cosets of subgroup H:

$$\begin{array}{rcl} \sigma_H : G & \longrightarrow & \mathrm{Sym}(G/H) \\ g & \mapsto & (\sigma_H(g) : Hx \mapsto Hxg) \quad \forall x \in G. \end{array}$$

This is a transitive representation but in general not faithful:

$$\ker(\sigma_H) = \bigcap_{g \in G} H^g = \mathrm{core}_G(H) \trianglelefteq G.$$

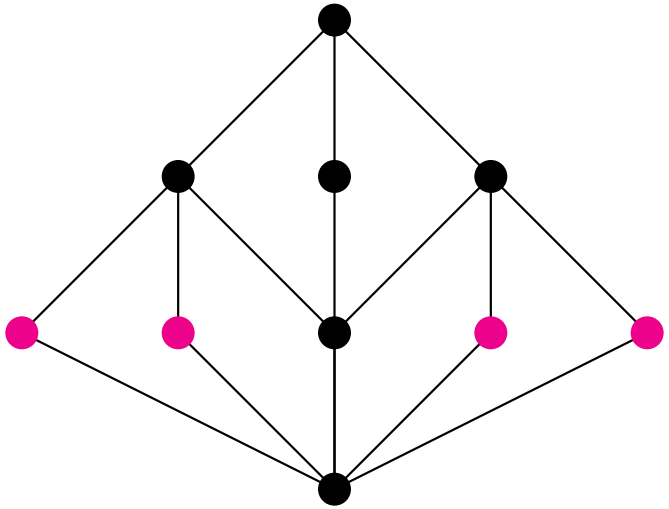
So can restate minimal degree as

$$\mu(G) = \left\{ \sum_{i=1}^{l} |G : H_i| \;\middle|\; \bigcap_{i=1}^{l} \mathrm{core}_G(H_i) = \{1\} \right\},$$

for some collection of subgroups $\{H_1, \ldots, H_l\}$.

Examples

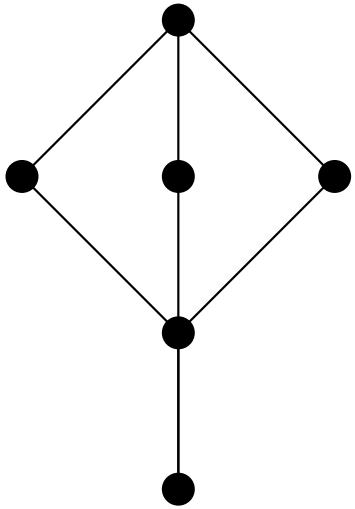
$$D_4 = \langle x, y \mid x^4 = y^2 = 1, x^y = x^{-1} \rangle$$



$$\mu(D_4) = \mu(C_4) = 4$$

Examples

$$Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, x^y = x^{-1} \rangle$$



$$\mu(Q_8) = 8$$

$\mu(G)$ is a funny invariant

Is usually not given by a transitive/primitive representation: it could have many orbits.
It is not additive over direct products in general:

$$\mu(G \times H) \neq \mu(G) + \mu(H)$$

Possible that there is a normal subgroup $N \trianglelefteq G$ such that

$$\mu(G/N) > \mu(G).$$

Such groups G are called exceptional with distinguished subgroup and quotient N and G/N respectively.

History of the exceptional groups problem

- Neumann (1985): Showed $\mu(G/N)$ can be exponentially larger than $\mu(G)$.

Neumann: *Algorithms for computing in permutation groups*

Let $G = D_4 \times \ldots \times D_4$. Then $\mu(G) = n\mu(D_4) = 4n$.
Let $N = \langle (z, z, 1, \ldots, 1), \ldots, (1, \ldots, 1, z, z) \,|\, z \in Z(D_4) \rangle$. Then

$$\mu(G/N) = 2^{n+1}.$$

example of exceptional permutation groups coming from central products.

History of the exceptional groups problem

- Easdown and Praeger (1988):
  - give many other examples of exceptional groups vis the central product construction
  - show $\mu(S \times T) = \mu(S) + \mu(T)$ for finite simple groups
  - no exceptional groups exist for 2-groups of order at most 2^4
  - classified exceptional 2-groups of order 2^5
    (doubted existence of exceptional groups of order p^5, p odd)
- Lemieux (1999, 2007)
  - argued no exceptional p-groups existed of order at most p^4
  - placed restrictions on possible structures of exceptional groups of order p^5:
    if $\mu(G/N) > \mu(G)$ with $|G| = p^5$, then $|N| = p$.

Conjecture (Abelian Quotients Conjecture (AQC))

Suppose G/N is abelian, then $\mu(G/N) \leq \mu(G)$.

- Kovaćs and Praeger (1989, 2000): AQC is true if:
  - G/N is elementary ableian
  - G does not contain an abelian normal subgroup
- The smallest counter-example must be a p-group (Frattini subgroup argument: Kovaćs-Praeger, Easdown)
- Franchi (2011): If G contains a maximal abelian subgroup, then $\mu(G/[G,G]) \leq \mu(G)$.
- Elias-Silbermann-TaklooBighash (2010): gave a greedy algorithm to calculate $\mu(G)$ for (nilpotent) groups

p-groups

Theorem (Britnell-S-Skyner, 2014)

- *Classification of exceptional groups of order p^5.*
- *As $p \to \infty$, the proportion of exceptional p-groups tends to $\frac{1}{2}$.*

Idea of the classification:

- Isolated groups of order p^4 that could give rise to exceptional permutation groups
- relied heavily on magma and the algorithm to compute exceptional groups for low odd primes

Prelude to the algorithm

From now on G will be a finite nilpotent (p)-group.

- Let $\mathcal{M}(G) = \{\text{minimal normal subgroups of } G\}$
- Let $\text{soc}(G)$ be the socle of G (vector space for finite p-groups)
- Let $\mathcal{T}(G) = \{T \lhd G \,|\, T \subseteq \text{soc}(G)\}$
  (lattice of normal subgroups contained in the socle)

For nilpotent groups, the condition about intersecting cores is essentially a condition about intersections contained in the socle of G, thus:

Definition

For H a subgroup of G, define

$$\text{rcore}(H) = \text{core}(H) \cap \text{soc}(G) = \langle N \in \mathcal{M}(G) \,|\, N \subset H \rangle.$$

Prelude to the algorithm

Since $\mathrm{soc}(G)$ is a vector space we can define:

Definition

For H a subgroup of G, define:

$$\begin{aligned} \dim_G(H) &:= \dim_{\mathrm{soc}(G)}(\mathrm{rcore}(H)) \\ \mathrm{codim}_G(H) &:= \dim(G) - \dim_{\mathrm{soc}(G)}(H) \end{aligned}$$

Definition (Socle Friendly Groups)

Call G socle friendly if for all subgroups H of G and all $T \in \mathcal{T}(G)$,

$$\mathrm{rcore}(H \cdot T) = \mathrm{rcore}(H) \cdot T.$$

Example

p-groups and nilpotent groups as socle friendly.

Prelude to the algorithm

Lemma (Replacement Lemma: Johnson, ESTB)

Let H be a subgroup of G with codimension at least 2. Then there exists two subgroups H_1 and H_2 such that

- $\mathrm{rcore}(H_1) \cap \mathrm{rcore}(H_2) = \mathrm{rcore}(H)$;
- $\frac{1}{|H_1|} + \frac{1}{|H_2|} \leq \frac{1}{|H|}$.
  (Equality $\iff$ G has two central involutions)

Corollary

- *We can always find a minimal faithful permutation representation of G consisting entirely of codimension-1 subgroups.*
- *If G is a p-group with p-odd, then all minimal faithful permutation representations consist of codimension-1 subgroups.*

The Algorithm

For $i \geq 0$ construct a pair $(\mathcal{R}_i, T_i)$ as follows:

- set $\mathcal{R}_0 = \varnothing$ and $T_0 = \mathrm{soc}(G)$

Given an $(\mathcal{R}_i, T_i)$ with $T_i \neq \{1\}$,

- find a subgroup H_{i+1} of G of maximal size not containing T_i
- set $\mathcal{R}_{i+1} = \mathcal{R}_i \cap H_{i+1}$ and $T_{i+1} = T_i \cap \mathrm{rcore}(H_{i+1})$
- if $T_i = \{1\}$, then set $\mathcal{R}_{i+1} = \mathcal{R}_i$ and $T_{i+1} = T_i$.

The sequence $(\mathcal{R}_i, T_i)$ is not unique, however:

Theorem (Elias-Silberman-TaklooBighash)

Let $d = \dim(G)$ with G socle friendly.

1. *For any choice of subgroups H_i, $T_{d-1} \neq \{1\}$;*
2. *$\mathcal{R}_d$ is a minimal faithful collection of size d;*
3. *Up to G-isomorphism, any minimal faithful collection of size d is obtained in this way.*

The Algorithm

Corollary (Johnson, Elias-Silberman-Takloo Bighash)

For p an odd prime:

- *The number of orbits in any minimal faithful representation is equal to* $\dim G$*, which also equals the number of generators of the centre* $Z(G)$*.*
- *If* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are two minimal faithful representations, then the sizes of the orbits coincide.*

Non-Socle-Friendly Group

Set $G = (V_1 \oplus V_2) \rtimes H$; where V_i are non-isomorphic 1-dimensional irreducible modules for H over $\mathbb{F}_q$. Then

$$\mu(G) = \frac{|H|q^2}{q} = |H|q,$$

whereas the algorithm yields $\frac{|H|q^2}{|H|q} + \frac{|H|q^2}{|H|q} = 2|H|q$.

Distinguished Quotients of p^5-Exceptional Permutation Groups

All possible distinguished quotients had order p^4.

Group	Presentation	Notes	Minimal Degree
Q_{16}	$\langle x, y \mid x^8 = 1,\ y^2 = x^4,\ [x,y] = x^{-2} \rangle$	$p = 2$	16
Q_{81}	$\langle x, y, z \mid x^9 = y^3 = z^3 = 1,\ [x,y] = 1,\ [x,z] = y,\ [y,z] = x^{-3} \rangle$	$p = 3$	27
$Q(p)$	$\langle x, y, z \mid x^{p^2} = y^p = z^p = [x,y] = [x,z] = 1,\ [y,z] = x^p \rangle$	p odd	p^3
$Q_1(p)$	$\langle x, y, z \mid x^{p^2} = y^p = 1,\ z^p = x^p,\ [x,y] = 1,\ [x,z] = y,\ [y,z] = x^p \rangle$	p odd	p^3
$Q_\alpha(p)$	$\langle x, y, z \mid x^{p^2} = y^p = 1,\ z^p = x^{\alpha p},\ [x,y] = 1,\ [x,z] = y,\ [y,z] = x^{\alpha p} \rangle$	p odd	p^3

Exceptional groups of order p^5 arranged by distinguished quotients.

For $p = 2$, the exceptional extensions have minimal degree 12; for $p \geq 3$, they have $2p^2$.

Group	Presentation	Notes
Q_{16}	$\langle x, y \mid x^8 = 1,\ y^2 = x^4,\ [x, y] = x^{-2} \rangle$	$p = 2$
G_1	$\langle x, y, n \mid x^8 = n^2 = 1,\ y^2 = x^4 n,\ n \text{ central},\ [x, y] = x^{-2} \rangle$	
G_2	$\langle x, y, n \mid x^8 = n^2 = 1,\ y^2 = x^4,\ n \text{ central},\ [x, y] = x^{-2} n \rangle$	
Q_{81}	$\langle x, y, z \mid x^9 = y^3 = z^3 = 1,\ [x, y] = 1,\ [x, z] = y,\ [y, z] = x^{-3} \rangle$	$p = 3$
G_3	$\langle x, y, z, n \mid x^9 = y^3 = z^3 = n^3 = 1,\ n \text{ central}, [x, y] = n,\ [x, z] = y,\ [y, z] = x^{-3} n \rangle$	
G_4	$\langle x, y, z, n \mid x^9 = y^3 = z^3 = n^3 = 1,\ n \text{ central},\ [x, y] = 1,\ [x, z] = y,\ [y, z] = x^{-3} n \rangle$	
$Q_1(3)$	$\langle x, y, z \mid x^9 = y^3 = 1,\ z^3 = x^3,\ [x, y] = 1,\ [x, z] = y,\ [y, z] = x^3 \rangle$	$p = 3$
G_4	$\langle x, y, z, n \mid x^9 = y^3 = n^3 = 1,\ n \text{ central},\ z^3 = x^3 n,\ [x, y] = n,\ [x, z] = y,\ [y, z] = x^3 n \rangle$	
G_5	$\langle x, y, z, n \mid x^9 = y^3 = n^3 = 1,\ n \text{ central},\ z^3 = x^3 n,\ [x, y],\ [x, z] = y,\ [y, z] = x^3 n^2 \rangle$	
G_6	$\langle x, y, z, n \mid x^9 = y^3 = n^3 = 1,\ n \text{ central},\ z^3 = x^3,\ [x, y] = 1,\ [x, z] = y,\ [y, z] = x^3 n \rangle$	
$Q_\alpha(3)$	$\langle x, y, z \mid x^9 = y^3 = 1,\ z^3 = x^6,\ [x, y] = 1,\ [x, z] = y,\ [y, z] = x^6 \rangle$	$p = 3,\ \alpha = 2$
G_6	$\langle x, y, z, n \mid x^9 = y^3 = n^3 = 1,\ n \text{ central},\ z^3 = x^6 n,\ [x, y] = 1,\ [x, z] = y,\ [y, z] = x^6 \rangle$	
G_7	$\langle x, y, z, n \mid x^9 = y^3 = n^3 = 1,\ n \text{ central},\ z^3 = x^6,\ [x, y] = 1,\ [x, z] = y,\ [y, z] = x^6 n \rangle$	

Exceptional groups of order p^5 arranged by distinguished quotients.

All extensions have minimal degree $2p^2$.

Group	Presentation	Notes
$Q(p)$	$\langle x,y,z \mid x^{p^2}=y^p=z^p=[x,y]=[x,z]=1,\ [y,z]=x^p\rangle$	p odd
E_1	$\langle x,y,z,n \mid x^{p^2}=y^p=z^p=1,\ n \text{ central},\ [x,y]=[x,z]=1,\ [y,z]=x^pn\rangle$	
E_2	$\langle x,y,z,n \mid x^{p^2}=z^p=1,\ y^p=n,\ n \text{ central},\ [x,y]=[x,z]=1,\ [y,z]=x^pn\rangle$	
E_3	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p,\ n \text{ central},\ [x,y]=1,\ [x,z]=n,\ [y,z]=x^pn\rangle$	
E_4	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=n,\ n \text{ central},\ [x,y]=1,\ [x,z]=n,\ [y,z]=x^pn\rangle$	
E_5	$\langle x,y,z,n \mid x^{p^2}=z^p=1,\ y^p=n,\ n \text{ central},\ [x,y]=1,\ [x,z]=n,\ [y,z]=x^p\rangle$	
$E_6(\lambda)$	$\langle x,y,z,n \mid x^{p^2}=z^p=1,\ y^p=n,\ n \text{ central},\ [x,y]=1,\ [x,z]=n^\lambda,\ [y,z]=x^pn\rangle$	$\lambda \neq 0,\ \left(\frac{1+4\lambda}{p}\right)=1$

Exceptional groups of order p^5 arranged by distinguished quotients.

All extensions have minimal degree $2p^2$.

Group	Presentation	Notes
$Q_1(p)$	$\langle x,y,z \mid x^{p^2}=y^p=1,\ z^p=x^p,\ [x,y]=1,\ [x,z]=y,\ [y,z]=x^p\rangle$	$p>3$
$F_1^{(1)}$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^p,\ n \text{ central},\ [x,y]=1,\ [x,z]=y,\ [y,z]=x^pn\rangle$	
$F_2^{(1)}$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^p,\ n \text{ central},\ [x,y]=n^{-1},\ [x,z]=y,\ [y,z]=x^pn^2\rangle$	
$F_3^{(1)}$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^pn,\ n \text{ central},\ [x,y]=1,\ [x,z]=y,\ [y,z]=x^pn\rangle$	
$F_4^{(1)}(\lambda)$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^pn,\ n \text{ central},\ [x,y]=n^{-1},\ [x,z]=y,\ [y,z]=x^pn^{1+\lambda}\rangle$	$\lambda\geq 0,\ \left(\frac{\lambda^2+4}{p}\right)=1$
$F_5^{(1)}(\lambda)$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^pn,\ n \text{ central},\ [x,y]=n^{-\alpha},\ [x,z]=y,\ [y,z]=x^pn^{\alpha+\lambda}\rangle$	$\lambda\geq 0,\ \left(\frac{\lambda^2+4\alpha}{p}\right)=1$
$Q_\alpha(p)$	$\langle x,y,z \mid x^{p^2}=y^p=1,\ z^p=x^{\alpha p},\ [x,y]=1,\ [x,z]=y,\ [y,z]=x^{\alpha p}\rangle$	$p>3$
$F_1^{(\alpha)}$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^{\alpha p},\ n \text{ central},\ [x,y]=1,\ [x,z]=y,\ [y,z]=x^{\alpha p}n\rangle$	
$F_2^{(\alpha)}$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^{\alpha p},\ n \text{ central},\ [x,y]=n^{-1},\ [x,z]=y,\ [y,z]=x^{\alpha p}n^{(\alpha+1)}\rangle$	
$F_3^{(\alpha)}$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^{\alpha p}n,\ n \text{ central},\ [x,y]=1,\ [x,z]=y,\ [y,z]=x^{\alpha p}n\rangle$	
$F_4^{(\alpha)}(\lambda)$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^{\alpha p}n,\ n \text{ central},\ [x,y]=n^{-1},\ [x,z]=y,\ [y,z]=x^{\alpha p}n^{\alpha+\lambda}\rangle$	$\lambda\geq 0,\ \left(\frac{\lambda^2+4\alpha}{p}\right)=1$
$F_5^{(\alpha)}(\lambda)$	$\langle x,y,z,n \mid x^{p^2}=y^p=1,\ z^p=x^{\alpha p}n,\ n \text{ central},\ [x,y]=n^{-\alpha},\ [x,z]=y,\ [y,z]=x^{\alpha p}n^{\alpha(\alpha+\lambda)}\rangle$	$\lambda\geq 0,\ \left(\frac{\lambda^2+4}{p}\right)=1$

Theorem (Isomorphisms of Exceptional Extensions)

For $p > 3$, the exceptional extensions of $Q_1(p)$ and $Q_\alpha(p)$ coincide:

- $F_1^{(1)} \cong F_1^{(\alpha)}$, $F_2^{(1)} \cong F_2^{(\alpha)}$ *and* $F_3^{(1)} \cong F_3^{(\alpha)}$;
- *each group in* $\{F_4^{(1)}(\lambda)\}$ *is isomorphic with a group in* $\{F_5^{(\alpha)}(\lambda)\}$; *while*
- *each group in* $\{F_5^{(1)}(\lambda)\}$ *is isomorphic with a group in* $\{F_4^{(\alpha)}(\lambda)\}$.

Theorem (Easdown-Praeger, 1987; Britnell-S-Skyner, 2014)

The total number of exceptional groups of order p^5 is:

$$\begin{cases} 2 & \text{if } p = 2, \\ 10 & \text{if } p = 3, \\ p+6 & \text{otherwise.} \end{cases}$$

Asymptotics

The number of groups of order p^5 is
$2p + 61 + (4, p-1) + 2(3, p-1)$

Corollary

The number of exceptional of order p^5 is asymptotically $\frac{1}{2}$.

What about larger p-groups?

There are

$$3p^2 + 39p + 344 + 24(3, p-1) + 11(4, p-1) + 2(5, p-1)$$

groups of order p^6.
Data seems to suggest proportion of exceptional groups goes to 0 and $p \to \infty$.

???Applications to Cohomology???

For G a p-group, define

$$\nu(G) = \min\{m \mid G \hookrightarrow \prod \mathrm{Sym}(p^m)\}.$$

Related to the largest orbit in a minimal faithful representation of G. Define

$$\begin{aligned} ee(G) &= \min\{m \mid p^m H^i(G,\mathbb{Z}) = 0,\ \forall\forall i\} \\ e(G) &= \min\{m \mid p^m H^i(G,\mathbb{Z}) = 0,\ \forall i\}. \end{aligned}$$

We have

$$ee(G) \leq e(G)$$

and

$$ee(G) \leq \nu(G) \leq \mu(G).$$