A generalization of the Z^*-theorem

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This is joint work with Ellen Henke.

Outline

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- Background

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- Fusion systems

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- Open questions

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- $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in Z(G)$ is isolated in G: $ijk \in Q_8$ is invariant under 3-fusion

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- When $p = 2$ for "known" involution centralizers without CFSG by Waldecker (2013)

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- $\mathcal{F}$ could include *all* injective maps between subgroups!
- $\mathcal{F}$ is *saturated* if morphisms extend and restrict in a way satisfied by morphisms in $\mathcal{F}_S(G)$

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- If $\mathcal{E} \subseteq \mathcal{F}$, x *centralizes* $\mathcal{E}$ if $\mathcal{E} \subseteq C_{\mathcal{F}}(x)$

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- But clearly $(4,5,6)$ does not centralize N

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Lemma

If $\varphi \in \mathrm{Aut}_G(P)$ is a p'-element with $[P, \varphi] \leqslant P \cap N$ and $\varphi|_{P\cap N} \in \mathrm{Aut}_N(P \cap N)$, then $\varphi \in \mathrm{Aut}_N(P)$.

- Proof uses conjugacy of Hall π-subgroups in soluble groups

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- Coprime action (using that P is $\mathcal{F}$-radical):

Proof outline

- May assume $O_{p'}(N) = 1$ Suppose t is N-isolated
- Let $G_0 = N\langle t\rangle$, $S_0 = (S \cap N)\langle t\rangle$
- By the Z^*-theorem, $Z(\mathcal{F}_{S_0}(G_0)) = Z(G_0) \leqslant C_S(N)$
- Hence just need $t \in Z(\mathcal{F}_{S_0}(G_0))$
- Assume P is $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully normalized
- Then enough to show $[t, \varphi] = 1$, $\varphi \in \mathrm{Aut}_{G_0}(P)$
- May assume φ has p'-order: $\mathrm{Aut}_{G_0}(P) = \mathrm{Aut}_{S_0}(P)O^p(\mathrm{Aut}_{G_0}(P))$
- By assumption $\varphi|_{P\cap N}$ extends to $\psi \in \mathcal{F}_S(G)$ fixing t
- Key lemma implies $\psi \in \mathcal{F}_{S\cap N}(N)$
- Coprime action (using that P is $\mathcal{F}$-radical):

$$t\varphi = t(\varphi\psi^{-1})(\psi) = t\psi = t$$

Open questions

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- Result implies there is a saturated fusion system:

$$\mathcal{F}_{C_S(N)} := \langle \mathrm{Inn}(C_S(N)), O^p(\mathrm{Aut}_{C_{\mathcal{F}}(T)}(P)) \mid P \leqslant C_S(N) \rangle$$

- $\mathcal{F}_{C_S(N)} = \mathcal{F}_{C_S(N)}(C_G(N))$?

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- $\mathcal{F}_{C_S(N)} = \mathcal{F}_{C_S(N)}(C_G(N))$?
- Generalization to compact lie groups?
- $C_S(\mathcal{N}) = C_S(\mathcal{E})$ when $\mathcal{N} \trianglelefteq \mathcal{L}$ are localities (in the sense of Chermak)?

Thank you!